Lecture 10: Turbulence: Transient or Sustained?

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Many realised in the 1980s that Chaos theory was not going to solve the problem of turbulence because it is an inherently spatiotemporal phenomenon (at least at low to intermediate Re). Pomeau [12][1] was the first to start thinking about turbulence from a statistical mechanical viewpoint by comparing spatiotemporal intermittency to directed percolation. Here, a spatial lattice of sites which individually can either be active ('turbulent') or passive ('laminar') is stochastically evolved in time using simple rules which incorporate information about neighbouring states. In the simplest models, there is one parameter p which defines the evolutionary strategy and then the challenge is to characterise the ensuing dynamics as a function of p. What typically emerges is that the order parameter $\rho(t)$ defined as the ensemble average of the lattice average of active sites (active=1, passive =0) asymptotes to 0 as $t \to \infty$ for $p \leq p_c$ where p_c is a critical value, whereas for $p > p_c$, $\lim_{t\to\infty} \rho(t) \sim (p-p_c)^\beta$ (there are universality classes defined by the exact value taken by the exponent β). What is important here is the idea that turbulence and the laminar state can coexist above a definite threshold (in Re) and the use of statistical techniques to characterise this via an order parameter (e.g. turbulent fraction in a domain).

These ideas were followed up most famously in 1998 by Bottin et al. [4] who conducted a series of plane Couette flow experiments in very large domains (non-dimensionalised as $380 \times 2 \times 70$ in the streamwise, cross-stream and spanwise directions respectively) so that the spatiotemporal behaviour near the transition threshold could be seen. The turbulent fraction of the flow, F_t as a function of time for various Re is shown in Figure 1. This plot emphasizes the temporal variability in F_t and the sensitivity of the flow to the initial conditions used (e.g. compare the two time signals for $Re = 322$). F_t is found to approach 0 eventually for all $Re < R_c \approx 323$, but long transients are found when $R_u \approx 312 < Re < R_c$ so that a long-time average which is non-zero can still be defined (all turbulence rapidly decays for $Re < R_u$). This is plotted verses Re in Figure 2.

Bottin et al. then applied a statistical approach to quantifying the transience of turbulence in the Re range $[R_u, R_c]$. They collected lifetime data from $50-120$ separate experiments and then estimated $P(T)$, the probability that the flow still remains turbulent after a time T: see Figure 3. The best fit lines drawn through the data at each Re indicate an exponential distribution of lifetimes

$$P(T) = e^{-T/\tau}, \tag{1}$$

[1] Apparently, this work was mostly done at the Woods Hole summer program of 1985.

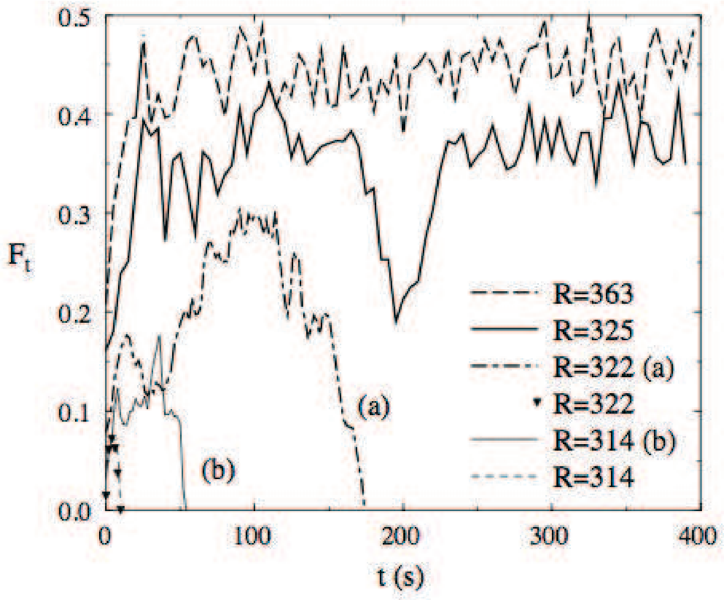

Figure 1: (From [4]) Turbulent fraction vs. time during typical runs at various values of Re.

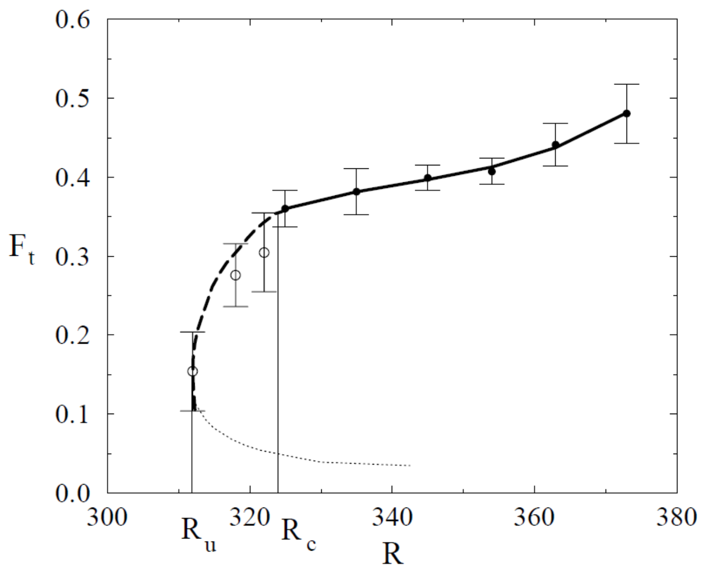

Figure 2: The time-averaged turbulent fraction against Re from [4]. $R_u = 312$ is the Re threshold below which turbulent patches rapidly decay (see Figure 1)). For $R_u < Re < R_c \approx 323$, there are long-lived turbulent transients. Above R_c, turbulence is sustained although since $F_t < 1$ it is not space-filling (the lower dotted line is a conjectured threshold).

(since $P(0) = 1$) where $\tau = \tau(Re)$ is the mean lifetime of the process. If $p(T)dT$ is the probability that the flow relaminarises in the time interval $[T, T + dT)$ (as $dT \to 0$), then

$$p(T) = \frac{1}{\tau} e^{-T/\tau} \tag{2}$$

and the half life (median) is $\tau \ln 2$. This distribution indicates that the relaminarisation process is *memoryless*, that is, the probability of relaminarising in the interval $[T, T + s)$ only depends on s and not T. Figure 3 also shows that flows at higher values of Re take longer to relaminarise. In fact plotting $1/\tau$ against Re indicated a linear relationship with an intercept ($\tau \to \infty$) at ≈ 323: see Figure 3. This is consistent with Re_c such that for $Re < Re_c$ the turbulence will always be transient with a finite half life, while for $Re > Re_c$ the half life is infinite and the turbulence sustained.

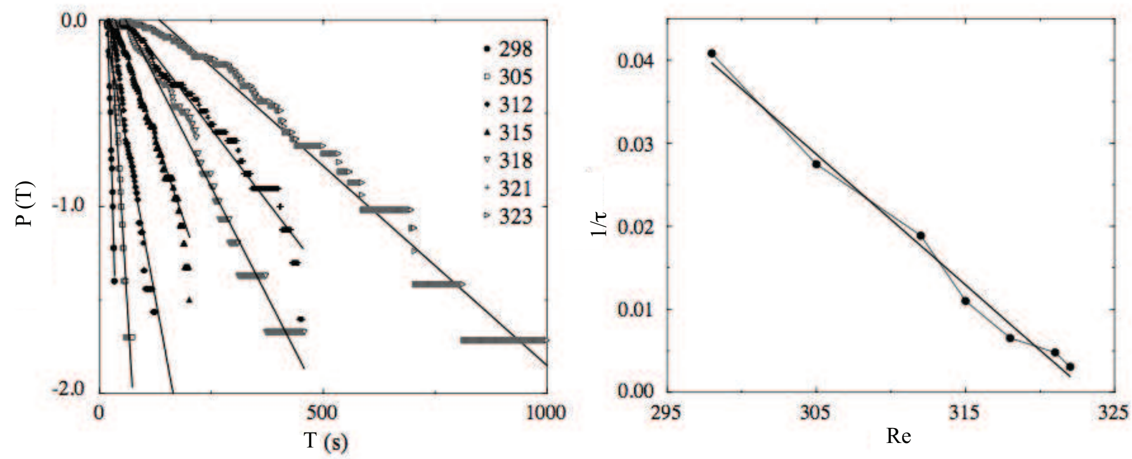

Figure 3: (From [4]) Left: cumulated lifetime distributions for turbulent transients at different values of $Re < Re_c$ indicating exponential decay (lin-log scales; solid lines are fits through the experimental data points). Right: variation of the inverse average decay time $1/\tau$ as a function of Re extrapolating to zero at $Re_c = 323$.

A similar statistical approach was also adopted numerically in *small* systems, that is, flow geometries where the flow is either globally laminar *or* turbulent: see Schmiegel & Eckhardt (1997) for plane Couette flow and Faisst & Eckhardt (2004) for pipe flow. The latter study was motivated by an experimental study by Daryshire & Mullin (1995) which showed no sharp border between initial conditions which lead to turbulence and those that did not. Faisst and Eckhardt found a similar situation when observing over a fixed period of time in their short, $5D$ (5 diameters) long pipe across which they applied periodic boundary conditions and through which they enforced constant mass flux. They collected lifetime statistics based on repeatedly initializing a numerical simulation using a perturbation of fixed form but randomly varying its amplitude. 50-100 different runs were done for each Re and 8 values of Re chosen from the interval $[1600, 2200]$. As in [4], they found that $P(T) = e^{-T/\tau(Re)}$ - see Figure 4 - and estimated τ using the half life rather than the mean lifetime due to the cut off in the observation times imposed. Figure 5 shows that the mean lifetime τ increases rapidly with Re with the inset figure indicating that actually $\tau \to \infty$ as $Re \to 2250$. Faisst and Eckhardt speculated that for $Re < 2250$ where the turbulent lifetime is finite, there is a chaotic repeller while for $Re > 2250$ there is a chaotic attractor.

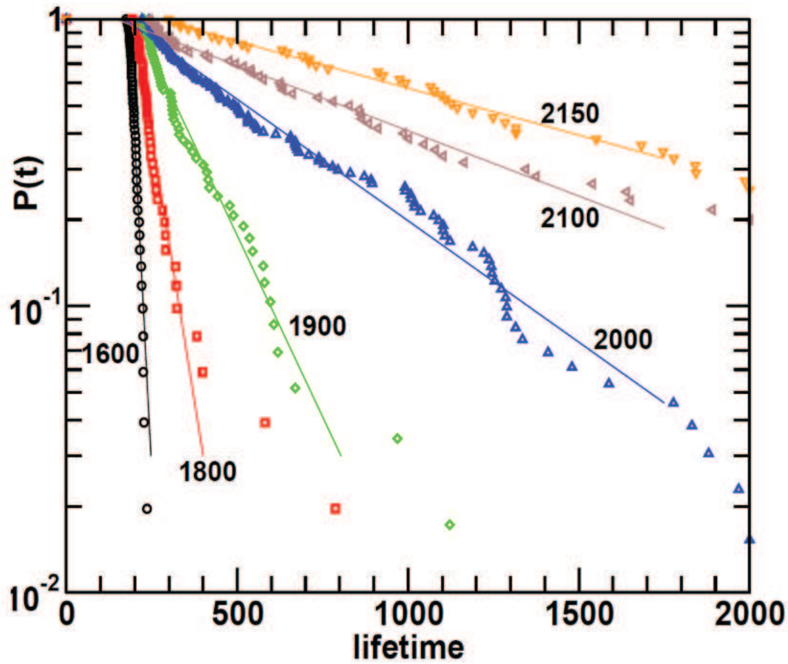

Figure 4: (From [6]) Probability for a single trajectory to still be turbulent after a time t for six Reynolds numbers as indicated.

Further evidence for critical point behaviour in the transition to turbulence in a pipe was presented by a novel experiment by Peixinho and Mullin in 2006 [11]. In their experimental setup (depicted in Figure 6a), a short duration perturbation was used to generate a localised puff at $Re = 1900$. The puff was allowed to advect $100D$ down the pipe so as to become independent of the initial conditions, at which point Re was lowered to the required value. The lifetime of the turbulence from this point onwards was then measured up to a maximum travel of $500D$ (their pipe was $785D$ long in total). The mean puff lifetime as a function of Reynolds number is shown in Figure 6b. In qualitative agreement with the conclusions of Faisst and Eckhardt's numerical simulations, the experiments showed that above a critical Reynolds number - estimated to be $Re_c \approx 1750 \pm 10$ - the lifetime of the puffs becomes infinite and turbulence is sustained.

An experimental and numerical study by Hof et al. in 2006 [8], however, failed to find any evidence for a critical Reynolds number. Instead, their results indicated that although the half life of turbulence increases rapidly with Re it never actually becomes infinite for finite Re so that pipe turbulence remains transient for *all* Re. Their experiments were performed using a longer (30m) and thinner (4mm diameter) *opaque* pipe which was non-dimensionally much longer at $7500D$ than Peixinho & Mullin's. Turbulent puffs were excited by injecting water through holes in the pipe (apparatus shown in Figure 7a). Since the pipe

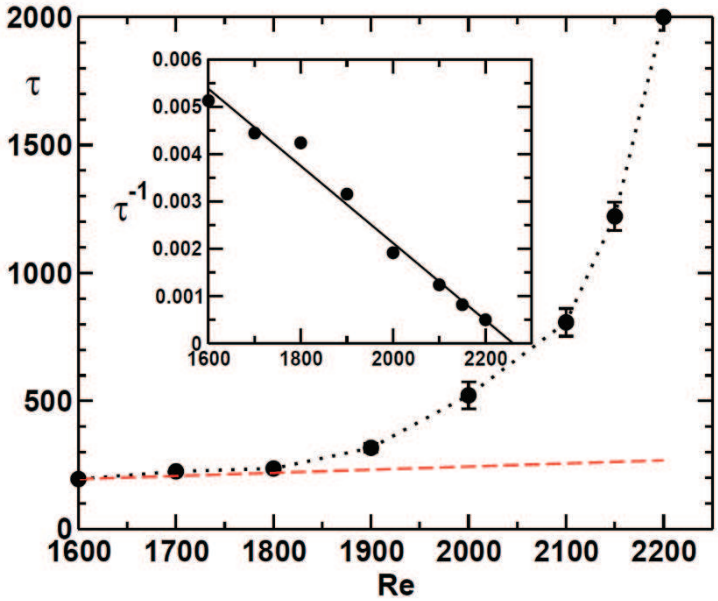

Figure 5: (From [6]) τ is found to increase rapidly with Re until the cut-off lifetime of 2000 at $Re = 2200$ is reached. The red dashed line shows the linear increas e in lifetime expected due to purely non-normal linear dynamics. The inset shows the inverse mean lifetime vs. Re and a linear fit, corresponding to a law $\tau(Re) \propto (Re_c - Re)^{-1}$, with $Re_c \approx 2250$.

was opaque, the angle at which the jet exited the pipe was monitored to see if the puffs had survived or not (an exiting puff causes a small flicker in the jet). This meant survival distances were measured rather than survival times with the latter found assuming that the puff speed is uniform. Other differences with the experiments of Peixinho & Mullin (2006) included using fixed-pressure-gradient driving rather than constant mass flux and, since the pipe was opaque, the implicit assumption that puffs were always triggered by the jets. These subtleties aside, their mean lifetime data (inset of Figure 7b) did not appear to fit the simple exponential implied by $\tau(Re) \propto (Re_c - Re)^{-1}$, as in [6] and [11]. Instead, they suggested an exponential relation between τ and Re, of the form

$$P(T; T_0) = e^{-(T-T_0)/\tau(Re)} \tag{3}$$

with lifetime given by

$$\frac{1}{\tau} = e^{(a+bRe)}$$

where a and b are constant fitting parameters. The significance of an exponential relationship is that there is no critical Reynolds number beyond which turbulent puffs are sustained for all times. The introduction of a shifted time origin T_0 was crucial in their data analysis: the best straight line fit needs to be shifted in time. The explanation for this was that the flow would take a finite time $\approx T_0$ to *reach* the turbulence state after the initial disturbance, so that $T - T_0$ would actually be the puff lifetime. Hof et al (2006) also redid the short pipe numerical computations of [6] to reach the same conclusion. Furthermore, they reprocessed

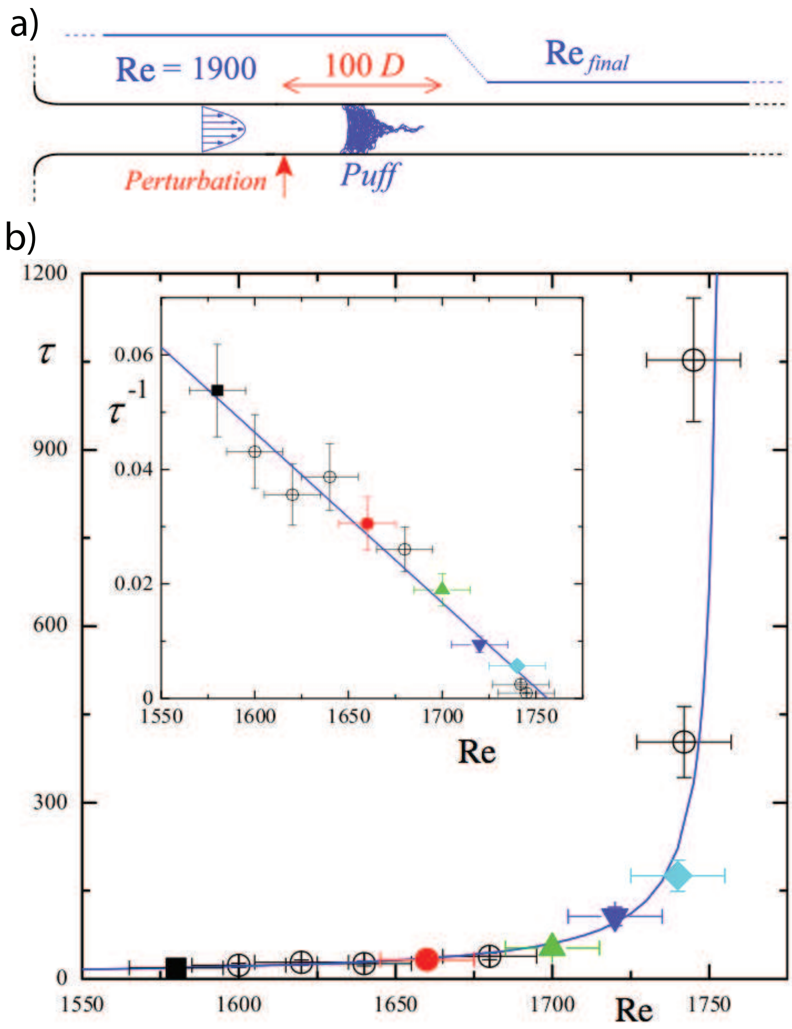

Figure 6: (From [11]) a) Schematic of flow control procedure. Laminar pipe flow was developed at $Re = 1900$ for $185D$ before a perturbation was injected (indicated by the arrow). The puff progressed downstream for $100D$ and Re was then reduced to a prescribed value. b) Variation of the mean lifetime as a function of Re and a fit, which indicates a sharp cutoff at $Re_c \approx 1750 \pm 10$. The inset is the inverse mean lifetime versus Re and a linear fit.

Faisst & Eckhardt's original data incorporating a best-fitted time origin to confirm that this data also supported a lack of a critical Re.

Next to attack this problem were Willis & Kerswell in 2007 [15] who carried out numerical simulations in a pipe long enough (10 times longer than in [6]) to realistically capture the localised structure of a turbulent puff. The methodology for generating initial conditions mirrored that of [11], as puffs at a higher Re of 1900 were used as initial conditions for the numerical simulations at lower Re. Using 40-60 independent simulations per Re as the computations were so costly, a critical Re_c was found with a value of ≈ 1870 and τ best

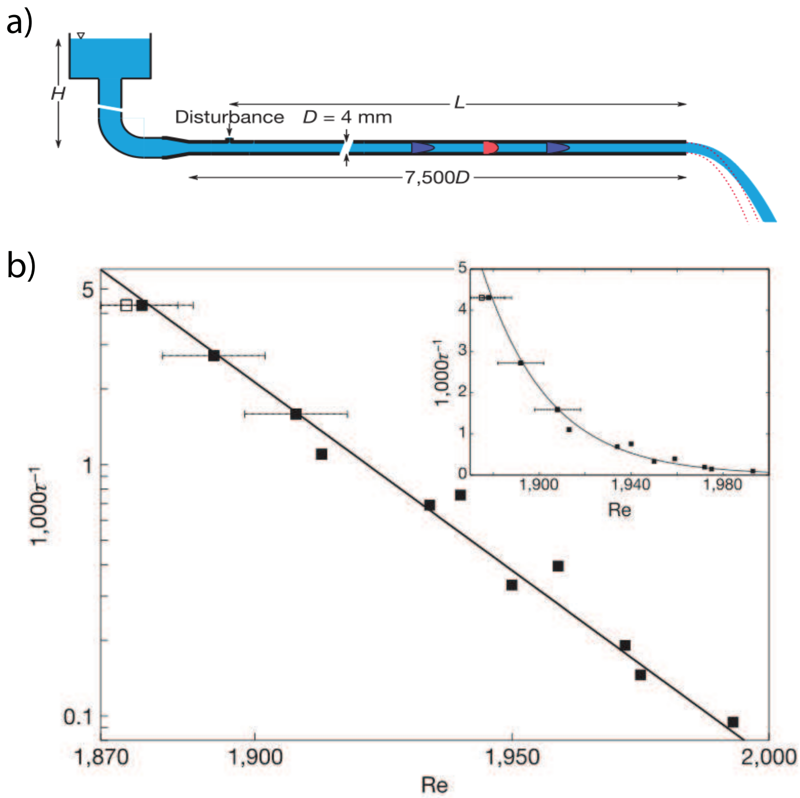

Figure 7: (From [8]) a) Sketch of the experimental apparatus. b) The reciprocal characteristic lifetimes as a function of Re for all experiments with $t_0 = 120D/U$, plotted on a log scale. Each data point required 400 to 500 measurements, with the total number of experiments underlying the figure exceeding 5,000. The straight line is an exponential fit to the data points. Inset, the same data replotted on a linear scale, underlining that they are not compatible with a diverging mean lifetime.

fitted by

$$\tau = \alpha\,(1870 - Re)^{-1}\,,$$

where $\alpha = 2.4 \times 10^{-4}$, comparing favourably with the results of [11] who found

$$\tau = \alpha\,(1750 - Re)^{-1}\,,$$

with $\alpha = 2.8 \times 10^{-4}$.

This result was countered by Hof et al. in 2008 [7] who presented results from four different physical experiments (pipe length $=600D$, $690D$, $2000D$ & $3600D$) in three different locations (Manchester, Delft and Göttingen) (see also the arXiv discussion articles arXiv:0707.2642 and arXiv.0707.2684). The authors also increased the number of observations taken compared to their previous paper [8]. From this greater data set, they revised their exponential dependence of τ upon Re to superexponential,

$$\tau \sim e^{\alpha Re} \Rightarrow \tau \sim e^{e^{\alpha Re}}.$$

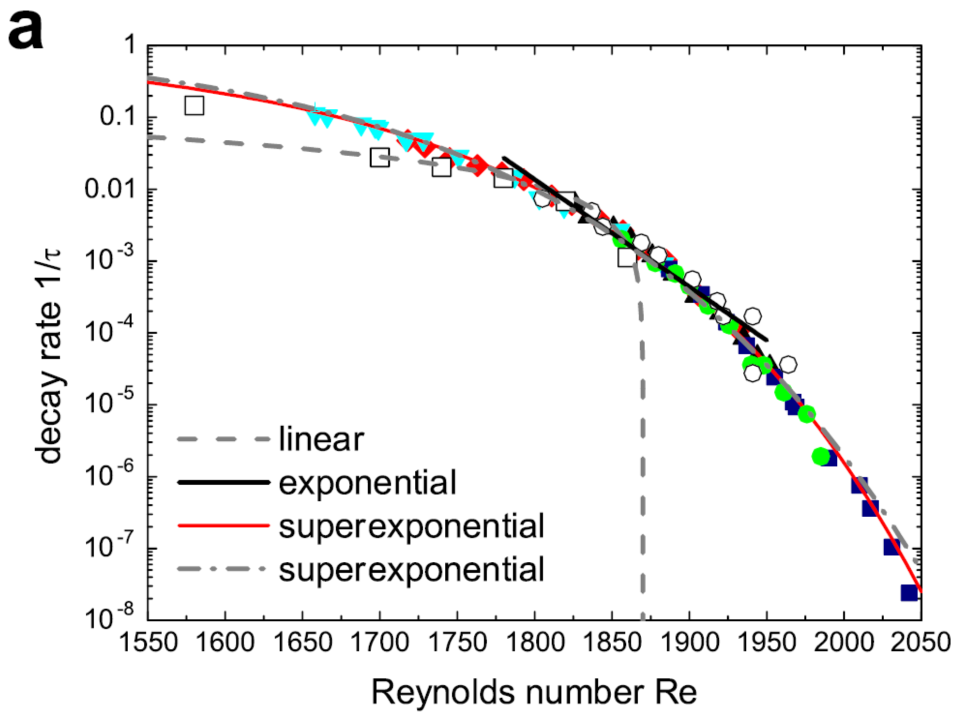

Figure 8: (From [7]) Decay rate $1/\tau$ plotted on a log linear scale against Re suggesting superexponential dependence of τ upon Re.

This new relationship still implied that no finite value for Re_c existed (see Figure 8: note the last Re considered and the later work of [1] discussed below).

Further numerical work then started to be done in large domains. Initially, due to the extreme cost, these simulations were carried out working with reduced resolution in one direction in order to ensure enough data was collected of long transients (e.g. Lagha & Manneville 2009, Willis & Kerswell 2009). The hope was that these reduced models would capture the real qualitative aspects of the problem, that is, whether there is a finite Re_c or not. In work by Lagha & Manneville [9] on plane Couette flow, the authors heavily reduced the resolution in the wall-normal direction, while maintaining the resolution in the two remaining directions. Using this approach they found evidence for a finite value for Re_c. Alongside this work, Willis & Kerswell [16] developed a similar reduction in pipe flow. Here the resolution reduction was made in the azimuthal direction, with just 3 Fourier modes retained in this direction ($m = 0, \pm 3$). Again the reduced model provided good qualitative comparisons with full DNS. Large numbers of simulations were carried out for both short and long pipes which suggested a transition in the relationship between τ and Re (Figure 9). For pipes too short to support localized turbulence, $Re_c = \infty$ with τ taking the form

$$\tau \sim e^{\alpha Re}.$$

When long pipes were studied, a different relationship emerged with a finite Re_c,

$$\tau \sim (Re_c - Re)^{\beta}.$$

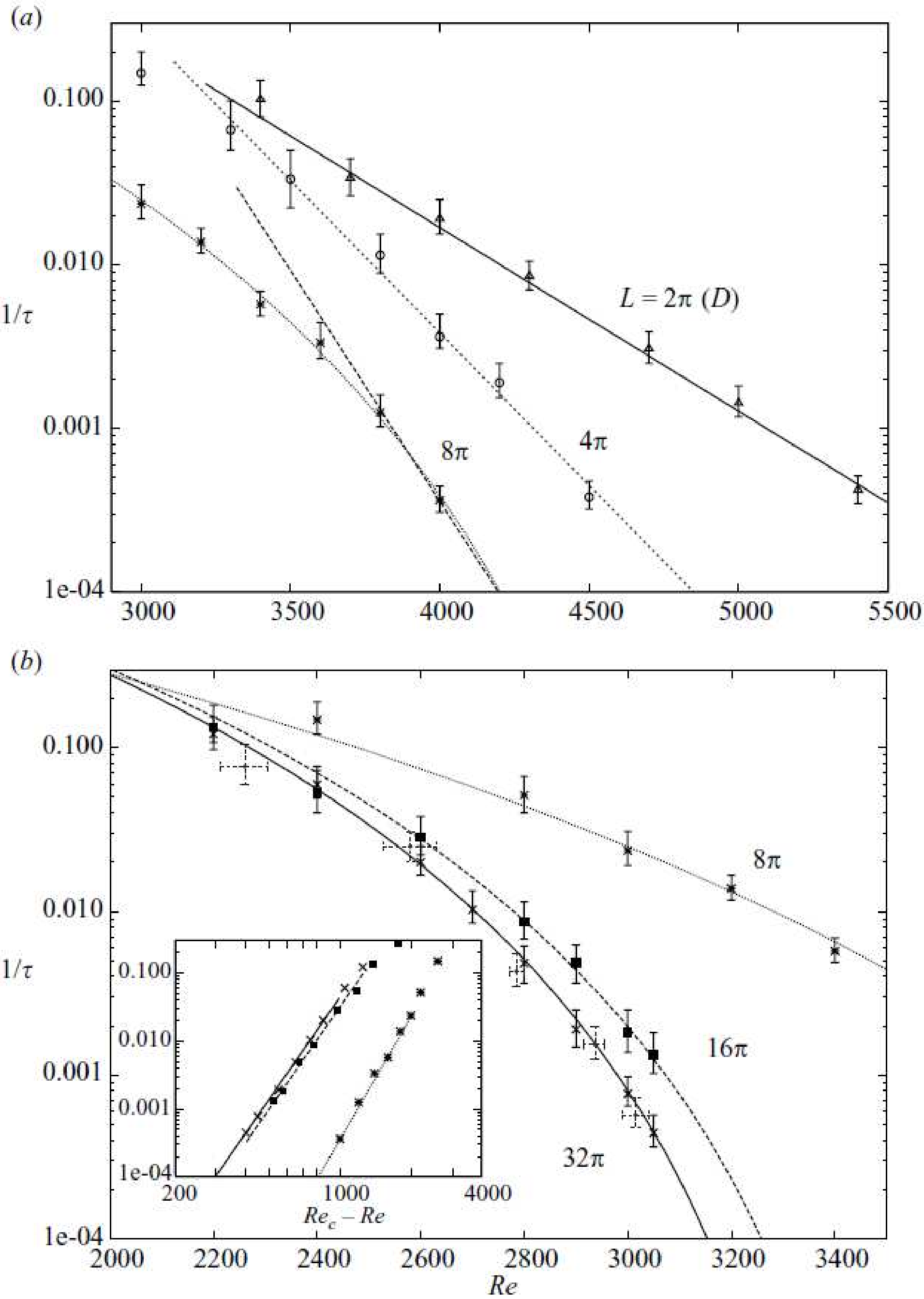

Figure 9: (From [16]) The sensitivity of lifetime (τ) to pipe length is shown with data generated using the $2+\epsilon$ dimensional model. The $1/\tau$ is plotted against Re for a range of pipe lengths from $2\pi\, D$ to $32\pi\, D \approx 100\, D$. The results suggest an infinite value of Re_c for pipes shorter than 8π and a finite value for longer pipes.

The '2+ϵ' dimensional model of [16] was so much more efficient to run that the full 3 dimensional situation that $100D$ pipes could easily be handled and transients followed for $O(100)$ times longer. However, this was not fully exploited because as Re was increased the puffs started to delocalise or split to form 'slugs' (a turbulent state which aggressively expands). This highlighted the fact that extrapolating puff lifetime data to asymptotically large Re was actually irrelevant since the morphology of the turbulence changes. It took a later study (Avila et al. 2011, see below) to pursue this realisation to a logical conclusion.

In the meantime, further numerical work was attempted in the $50D$ pipe to collect even more data by fully harnessing a supercomputer. Using the same numerical code as in [15] with the same resolution, Avila et al [2] extended the results of [15] in Re, sample size and included pipes of length $100\,D$. Armed with much more data, the authors saw no statistical evidence for Re_c being finite within their range of $Re < 1900$ (Figure 10).

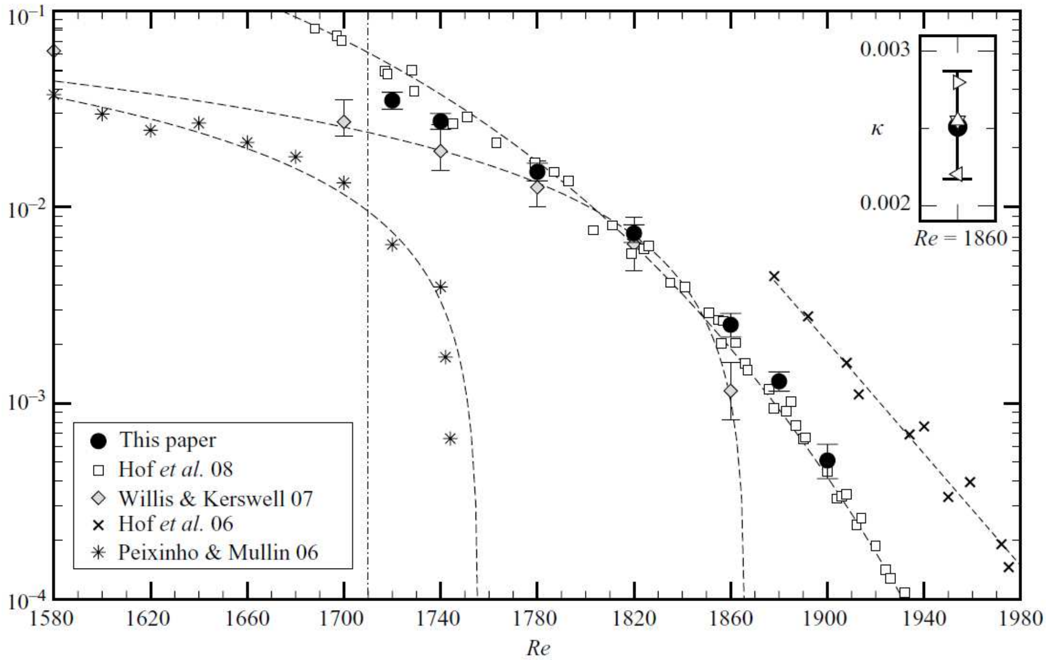

Figure 10: (From [2]) $1/\tau$ plotted against Re, summarising the recent work against their results which suggest a superexponential dependence of τ upon Re.

After this, Avila et al. 2011 [1] took measurements of a puff splitting in both numerical simulations and experimental work. They measured the lifetime of a puff before it underwent its first split to become two puffs, and calculated $S\,(T)$, the probability that a puff has not split by time T. Their results suggested this probability had an exponential (memoryless) form with a mean lifetime τ_s having a superexponential dependence on Re (Figure 11). By combining the plots of decay and splitting to see the crossover, a critical Reynolds number of 2040 was found beyond which, *on average*, a puff should survive. For Reynolds number smaller than this, puffs are more likely to decay than split, and therefore, on average to ultimately decay. In other words, below 2040 turbulence is transient, and above it, the expectation is that it will be sustained. Individual initial conditions at Reynolds numbers greater than 2040 can still lead to transient turbulence, but the expectation over an ensemble of runs is that more will yield turbulence at the end of a given time period (however long) than not.

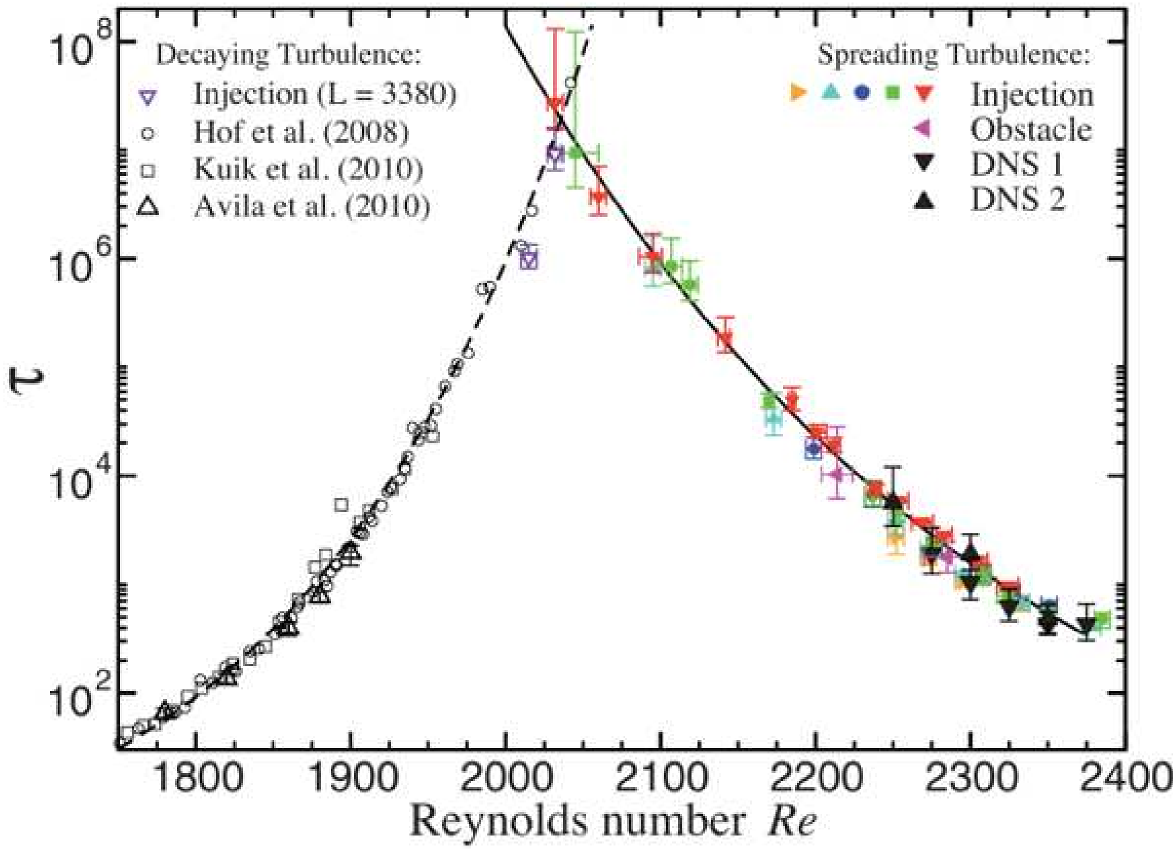

Figure 11: (From [1]) Mean lifetime before decay and the mean lifetime before splitting are plotted against Re (both experimental and numerical data shown).

The current conclusion is then the following. For small systems, all current evidence is that turbulence appears transient albeit with very large half life as Re increases. In large domains, however, the balance of evidence is that turbulence is ultimately sustained. The key difference between small and large systems is that in large systems, turbulent patches can independently exist in the flow. The spatial coupling between these turbulent patches appears crucial to achieve sustenance [10]. This realisation has led to a return of statistical approaches to modelling turbulence, most recently in the form of directed percolation [14] and other reduced models [3].

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