Bounds on Surface Stress Driven Flows

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January 12, 2009

1 Introduction

In the past fifteen years the background method of Constantin, Doering, and Hopf has been used to derive rigorous bounds on transport in fluid mechanics equations. The background method is based on the decomposition of the velocity or temperature field into a background field and a fluctuation field. The background field is specified so that the fluctuation field satisfies a partial differential equation with homogeneous boundary conditions. Many bounding problems have a variational formulation under the background method. The background method has been used to prove bounds on the mechanical dissipation rate in various driven turbulent flows. This work was started by Constantin and Doering [4]. They studied the mechanical dissipation in shear flows at high Reynolds number. Since then there has been a large number of papers about dissipation in turbulent shear flow with different boundary conditions and geometries.

We extend this technique to flows driven by stress at the boundary, namely Marangoni convection and surface shear stress driven flow. In the case of surface shear stress driven flow we study the energy stability of the laminar flow solution as a function of the Grashoff number and prove an upper bound on the friction coefficient for high Reynolds number. Tang, Caulfield, and Young [6] first used the background method to prove bounds for this type of problem and we compare our results to theirs.

The problem of Marangoni convection is related to that of stress driven shear flow. In the Marangoni case the surface stress is due to surface tension gradients. Pearson [5] developed the linear theory for Marangoni convection in 1958, and Davis [3] used variational methods to study the nonlinear stability problem in 1969. In the case of infinite Prandtl number we use nonvariational methods to improve the estimate of the critical Marangoni number for nonlinear stability of the conduction solution. We also use the background method to prove an upper bound on the Nusselt number in Marangoni convection.

2 Surface Stress Driven Flow

We consider the problem of flow in the two dimensional domain in Figure 2 subject to periodic boundary conditions in the horizontal x direction, no slip conditions at $z = 0$, and

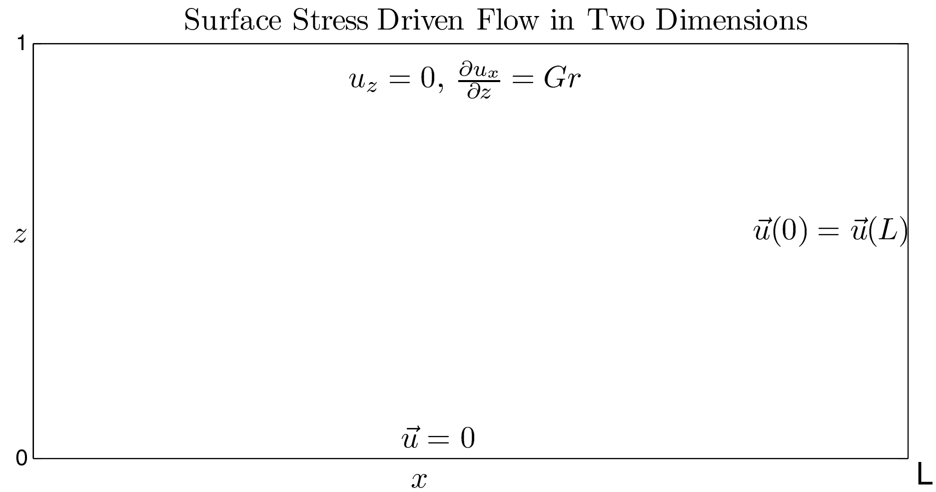


Figure 1: Domain for the surface stress driven flow.

a fixed shear stress at $z = h$. The full system of equations is:

$$\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = \nu \Delta u \tag{1}$$

$$\nabla \cdot u = 0 \tag{2}$$

$$u|_{z=0} = 0 \tag{3}$$

$$u_z|_{z=h} = 0 \tag{4}$$

$$\nu \frac{\partial u_x}{\partial z} = \tau \tag{5}$$

We take periodic boundary conditions in x, with a domain length L.

2.1 Energy Stability Analysis in Two Dimensions

The laminar flow solution of these equations is $u_x = \frac{\tau z}{\nu}$. We perform the energy stability analysis of this solution by making the substitution $u = U + \frac{\tau z}{\nu}\hat{x}$. The resulting system of equations is:

$$\frac{\partial U}{\partial t} + U \cdot \nabla U + \frac{\tau z}{\nu}\frac{\partial U}{\partial x} + U_z \frac{\tau}{\nu}\hat{x} + \nabla p = \nu \Delta U \tag{6}$$

$$\nabla \cdot U = 0 \tag{7}$$

$$U_x|_{z=0} = 0 \tag{8}$$

$$\frac{\partial U_x}{\partial z}|_{z=h} = 0 \tag{9}$$

$$U_z|_{z=0,1} = 0 \tag{10}$$

Take the dot product of the momentum equation with U and integrate over the domain to derive the energy expression:

$$\frac{1}{2}\frac{d\|U\|^2}{dt} + \int \frac{\tau}{\nu} U_x U_z dxdz = -\nu \|\nabla U\|^2 \tag{11}$$

If $\frac{d\|U\|^2}{dt} < 0$ for all U satisfying the perturbation's boundary conditions then the base solution is stable. This derivative is given by a quadratic form $-Q$ in U:

$$Q = \nu\|\nabla U\|^2 + \int \frac{\tau}{\nu} U_x U_z dxdz \tag{12}$$

We use variational methods to minimize Q, subject to the constraints $\nabla \cdot U = 0$ and $\|U\|^2 = 1$. The resulting Euler-Lagrange equations are:

$$\lambda U_x = -\nu \Delta U_x + \frac{\partial q}{\partial x} + \frac{\tau}{2\nu} U_z \tag{13}$$

$$\lambda U_z = -\nu \Delta U_z + \frac{\partial q}{\partial z} + \frac{\tau}{2\nu} U_x \tag{14}$$

$$\nabla \cdot U = 0 \tag{15}$$

$$\|U\|^2 = 1 \tag{16}$$

Here q is the Lagrange multiplier associated with incompressibility and λ is the Lagrange multiplier associated with the normalization condition. We non-dimensionalize the equations by making the substitutions $x' = \frac{x}{h}$ and $z' = \frac{z}{h}$. Define the Grashoff number $Gr = \frac{\tau h^2}{\nu^2}$. Then if $U'(x', z') = U(x, z)$ we have the following equations:

$$\lambda U_x = -\Delta U_x + \frac{Gr}{2} U_z + \frac{\partial q}{\partial x} \tag{17}$$

$$\lambda U_z = -\Delta U_z + \frac{Gr}{2} U_x + \frac{\partial q}{\partial z} \tag{18}$$

$$\tag{19}$$

If the smallest eigenvalue $\lambda_{min} > 0$ then Q is positive definite and the base solution is stable. Since the system is two dimensional we introduce the stream function Ψ, which satisfies $\frac{\partial \Psi}{\partial x} = U_z$ and $\frac{\partial \Psi}{\partial z} = -U_x$. Take linear combinations of the x and z derivatives of the resulting equations to eliminate the pressure. The Euler Lagrange equations become a fourth order equation:

$$\lambda \Delta \Psi = -\Delta^2 \Psi - Gr \frac{\partial^2 \Psi}{\partial x \partial z} \tag{20}$$

$$\frac{\partial \Psi}{\partial x}|_{z=0,1} = 0 \tag{21}$$

$$\frac{\partial \Psi}{\partial z}|_{z=0} = 0 \tag{22}$$

$$\frac{\partial^2 \Psi}{\partial z^2}|_{z=1} = 0 \tag{23}$$

$$\tag{24}$$

The system is periodic in x so we write Ψ in terms of its Fourier series. Let $k = \pi j/L$ be the wavenumber. We write $\Psi = \sum_k \hat{\Psi}_k e^{\imath kx}$. Then the equations become:

$$\lambda\left(\frac{\partial^2\hat{\Psi}}{\partial z^2} - k^2\hat{\Psi}\right) = -\left(\frac{\partial^4\hat{\Psi}}{\partial z^4} - 2k^2\frac{\partial^2\hat{\Psi}}{\partial z^2} + k^4\hat{\Psi}\right) - \imath Grk\frac{\partial\hat{\Psi}}{\partial z} \tag{25}$$

$$\hat{\Psi}|_{z=0,1} = 0 \tag{26}$$

$$\frac{\partial\hat{\Psi}}{\partial z}|_{z=0} = 0 \tag{27}$$

$$\frac{\partial^2\hat{\Psi}}{\partial z^2}|_{z=1} = 0 \tag{28}$$

We have supressed the k dependence of $\hat{\Psi}_k$. We search for the critical Grashoff number below which all of the eigenvalues λ are positive. This can be done with numerical methods. Each Fourier mode $\hat{\Psi}$ satisfies a fourth order ODE in z. We used finite differences to calculate the smallest eigenvalue corresponding to each k and determined the critical Grashoff number from this. After discretizing in z the differential equation becomes a generalized eigenvalue problem. We find that the critical Grashoff number is $Gr = 140$, at a value of $k = 3.1$.

2.2 Energy Stability for a Three Dimensional Stress Driven Flow

Consider the following non-dimensionalized equations for a three dimensional stress driven flow:

$$\frac{\partial u}{\partial t} + u\cdot\nabla u + \nabla p = \Delta u \tag{29}$$

$$\nabla\cdot u = 0 \tag{30}$$

$$u_z|_{z=0,1} = 0 \tag{31}$$

$$u_x|_{z=0} = 0 \tag{32}$$

$$u_y|_{z=0} = 0 \tag{33}$$

$$\frac{\partial u_x}{\partial z}|_{z=1} = Gr \tag{34}$$

$$\frac{\partial u_y}{\partial z}|_{z=1} = 0 \tag{35}$$

The domain is periodic in x and y. A steady solution to this equation is $u_x = Grz$. If we make the substitution $U = u + \hat{\imath}Grz$ we can perform the same analysis as in the two dimensional case. The variational formulation is different and so is the set of Euler-Lagrange equations:

$$\lambda U_x = -\Delta U_x + \frac{Gr}{2}U_z + \frac{\partial q}{\partial x} \tag{36}$$

$$\lambda U_y = -\Delta U_y + \frac{\partial q}{\partial y} \tag{37}$$

$$\lambda U_z = -\Delta U_z + \frac{Gr}{2}U_x + \frac{\partial q}{\partial z} \tag{38}$$

If we assume that the eigenfuction for the lowest eigenvalue is not a function of x, we can introduce the stream function Ψ defined by $\frac{\partial \Psi}{\partial y} = U_z$ and $\frac{\partial \Psi}{\partial z} = -U_y$. The lowest frequency eigenmodes of shear driven flows tend to be Langmuir circulation flows that are independent of the streamwise direction. Our use of this assumption is justified by empirical observations within this field [6]. The Euler Lagrange equations become:

$$\lambda U_x = \frac{Gr}{2}\frac{\partial \Psi}{\partial y} - \Delta U_x \tag{39}$$

$$-\lambda \frac{\partial \Psi}{\partial z} = \frac{\partial q}{\partial y} + \Delta \frac{\partial \Psi}{\partial z} \tag{40}$$

$$\lambda \frac{\partial \Psi}{\partial y} = \frac{\partial q}{\partial z} - \Delta \frac{\partial \Psi}{\partial y} + \frac{Gr}{2} U_x \tag{41}$$

Taking the z derivative with respect to the second expression and the y derivative with respect to the first and third we can subtract the last two to eliminate the pressure:

$$\lambda \frac{\partial U_x}{\partial y} = \frac{Gr}{2}\frac{\partial^2 \Psi}{\partial y^2} - \Delta \frac{\partial U_x}{\partial y} \tag{42}$$

$$\lambda \Delta \Psi = -\Delta^2 \Psi + \frac{Gr}{2}\frac{\partial U_x}{\partial y} \tag{43}$$

We can write an ODE for the fourier modes of Ψ:

$$\lambda \imath k U_x = -\frac{Gr}{2} k^2 \hat{\Psi} - \imath k \left(\frac{\partial^2 U_x}{\partial z^2} - k^2 U_x \right) \tag{44}$$

$$\lambda \left(\frac{\partial^2 \hat{\Psi}}{\partial z^2} - k^2 \hat{\Psi} \right) = -\left(\frac{\partial^4 \hat{\Psi}}{\partial z^4} - 2k^2 \frac{\partial^2 \hat{\Psi}}{\partial z^2} + k^4 \hat{\Psi} \right) + \frac{Gr}{2} \imath k U_x \tag{45}$$

$$\hat{\Psi}|_{z=0,1} = 0 \tag{46}$$

$$\frac{\partial \hat{\Psi}}{\partial z}|_{z=0} = 0 \tag{47}$$

$$\frac{\partial^2 \hat{\Psi}}{\partial z^2}|_{z=1} = 0 \tag{48}$$

$$U_x|_{z=0} = 0 \tag{49}$$

$$\frac{\partial U_x}{\partial z}|_{z=1} = 0 \tag{50}$$

This is an ODE and we can discretize it and convert it into a generalized eigenproblem. We look for the critical Grashoff number Gr where the system first has a negative eigenvalue for some k. We find that $Gr_c = 51.7$ at $k = 2.1$. This is in agreement with work done by Tang, Caulfield, and Young [6].

2.3 Bounds on the Energy Dissipation in Two Dimensions

Define $\langle \cdot \rangle$ to be the space time average and $\bar{\cdot}$ to be the horizontal and time average. Begin by taking the dot product of the momentum equation with u. After averaging we find that:

$$\nu \left\langle |\nabla u|^2 \right\rangle = \frac{\tau}{h} \bar{u}_x(h) \tag{51}$$

Define the dissipation $\epsilon = \langle \nu |\nabla u|^2 \rangle$, Reynolds number of the flow to be $Re = \frac{\bar{u}_x(h)h}{\nu}$, and the friction coefficient to be $C_f = \frac{\epsilon h}{\bar{u}_x(h)^3}$. For the steady state solution $C_f(Re) = \frac{1}{Re}$. We want to determine a limit on how C_f might scale with the Reynolds number by proving bounds on the mean horizontal velocity at the $z = h$. Introduce a background field horizontal velocity $U(z)$ such that U satisfies the same boundary conditions as u. Consider the decomposition $u = \hat{i}U + \tilde{u}$. The following are the dimensionless equations for $\tilde{u}$:

$$\frac{\partial \tilde{u}}{\partial t} + \tilde{u} \cdot \nabla \tilde{u} + \nabla p + U \frac{\partial \tilde{u}}{\partial x} + \hat{i} \tilde{u}_z \frac{dU}{dz} = \Delta \tilde{u} + \hat{i} \frac{d^2 U}{dz^2} \tag{52}$$

$$\tilde{u}|_{z=0} = 0 \tag{53}$$

$$\tilde{u}_z|_{z=h} = 0 \tag{54}$$

$$\frac{\partial \tilde{u}_x}{\partial z}|_{z=h} = 0 \tag{55}$$

Take the dot product with $\tilde{u}$ and compute the space time average:

$$0 = -\left\langle |\nabla \tilde{u}|^2 \right\rangle - \left\langle \frac{dU}{dz} \tilde{u}_x \tilde{u}_z \right\rangle + \left\langle \tilde{u}_x \frac{d^2 U}{dz^2} \right\rangle \tag{56}$$

The time derivative term vanishes because $\|\tilde{u}\|^2$ is bounded. If we integrate the last term by parts in the z direction the previous expression becomes:

$$0 = -\left\langle |\nabla \tilde{u}|^2 \right\rangle - \left\langle \frac{dU}{dz} \tilde{u}_x \tilde{u}_z \right\rangle + Gr \bar{\tilde{u}}_x(1) - \left\langle \frac{\partial \tilde{u}_x}{\partial z} \frac{dU}{dz} \right\rangle \tag{57}$$

$$= -\left\langle |\nabla \tilde{u}|^2 \right\rangle - \left\langle \frac{dU}{dz} \tilde{u}_x \tilde{u}_z \right\rangle + Gr(\bar{u}_x(1) - U(1)) - \left\langle \frac{\partial \tilde{u}_x}{\partial z} \frac{dU}{dz} \right\rangle \tag{58}$$

Substitute $\hat{i}U(z) + \bar{\tilde{u}}(z) = \bar{u}(z)$ into $\left\langle |\nabla u|^2 \right\rangle$:

$$\frac{1}{2} \left\langle |\nabla u|^2 \right\rangle = \frac{1}{2} \left\langle |\nabla \tilde{u}|^2 \right\rangle + \frac{1}{2} \left\langle \frac{dU}{dz}^2 \right\rangle + \left\langle \frac{\partial \tilde{u}_x}{\partial z} \frac{dU}{dz} \right\rangle \tag{59}$$

We take a linear combination of these expressions so as to eliminate the $\left\langle \frac{\partial \tilde{u}_x}{\partial z} \frac{dU}{dz} \right\rangle$ term:

$$\frac{1}{2} \left\langle |\nabla u|^2 \right\rangle = -\left\langle \frac{1}{2} |\nabla \tilde{u}|^2 + \frac{dU}{dz} \tilde{u}_x \tilde{u}_z \right\rangle + Gr(\bar{u}_x(1) - U(1)) + \frac{1}{2} \left\langle \frac{dU}{dz}^2 \right\rangle \tag{60}$$

Now we define the quadratic form $Q_U(\tilde{u})$:

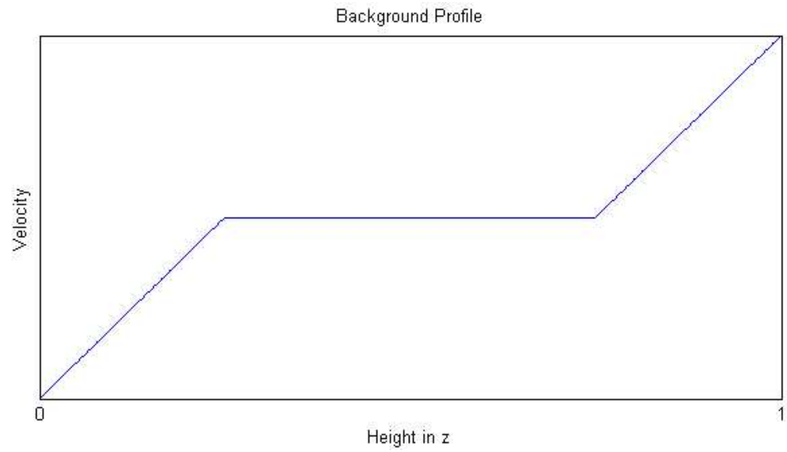
$$Q_U(\tilde{U}) = \left\langle \frac{1}{2} |\nabla \tilde{u}|^2 + \frac{dU}{dz} \tilde{u_x} \tilde{u_z} \right\rangle \tag{61}$$

Finally we combine this definition with the identity $\left\langle |\nabla u|^2 \right\rangle = Gr \bar{u}_x(1)$ to produce:

$$\bar{u_x}(1) = 2U(1) - \frac{1}{Gr} \left\langle \frac{dU}{dz}^2 \right\rangle + \frac{2}{Gr} Q \tag{62}$$

If we choose U so that Q is positive definite we arrive at the following bound for $\bar{u}_x(1)$:

$$\bar{u}_x(1) \geq 2U(1) - \frac{1}{Gr} \left\langle \frac{dU}{dz}^2 \right\rangle \tag{63}$$



Define the background horizontal velocity profile to be linear in z in horizontal boundary layers near the top and bottom boundary and constant elsewhere:

$$U(z) = Grz \qquad 0 < z < \delta_1 \tag{64}$$
$$= Gr\delta_1 \qquad \delta_1 < z < 1-\delta_2 \tag{65}$$
$$= Gr(\delta_1+\delta_2+z-1) \qquad 1-\delta_2 < z < 1 \tag{66}$$

Then $U(1) = Gr(\delta_1+\delta_2)$ and $\bar{u}_x(1) \geq Gr(\delta_1+\delta_2)$. We need to choose δ_1 and δ_2 to maximize the sum while keeping Q positive definite. We drop the accents and refer to the fluctuations away from the background as u.

$$Q = \frac{1}{2}\|\nabla u\|^2 + Gr\int_0^{\delta_1} u_x u_z dz dx + Gr\int_{1-\delta_2}^{1} u_x u_z dx dz \tag{67}$$

We start with the second term on the right hand side:

$$\int\int_0^{\delta_1} u_x u_z dz dx = \int\int_0^{\delta_1}\int_0^z \left(\frac{\partial u_x}{\partial z'}u_z + u_x\frac{\partial u_z'}{\partial z'}\right) dz' dx dz \tag{68}$$

Now we use incompressibility to eliminate the $u_x\frac{\partial u_z'}{\partial z'} = -u_x\frac{\partial u_x}{\partial x} = -\frac{\partial\|u_x\|^2}{\partial x}$ term. Then we find:

$$\left|\int\int_0^{\delta_1} u_x u_z dz dx\right| = \left|\int_0^{\delta_1}\int_0^z \left(\frac{\partial u_x}{\partial z'}\int_0^{z'}\frac{\partial u_z''}{\partial z''}dz''\right) dz' dx dz\right| \tag{69}$$
$$\leq \int\int_0^{\delta_1}\int_0^z \left|\frac{\partial u_x}{\partial z'}\right| \sqrt{z'}\|\frac{\partial u_z}{\partial z}\|_{\delta_1,z} dz' dx dz \tag{70}$$
$$\leq \int\int_0^{\delta_1} \frac{z}{\sqrt{2}}\|\frac{\partial u_x}{\partial z}\|_{\delta_1,z}\|\frac{\partial u_z}{\partial z}\|_{\delta_1,z} dx dz \tag{71}$$
$$\leq \frac{\delta_1^2}{4\sqrt{2}}\left(\frac{1}{C}\|\frac{\partial u_x}{\partial z}\|_{\delta_1}^2 + \frac{C}{2}\|\frac{\partial u_x}{\partial x}\|_{\delta_1}^2 + \frac{C}{2}\|\frac{\partial u_z}{\partial z}\|_{\delta_1}^2\right) \tag{72}$$

Here $\|\cdot\|_{\delta_1} = \left(\int\int_0^{\delta_1}\cdot^2 dxdz\right)^{1/2}$ and $\|\cdot\|_{1-\delta_2} = \left(\int\int_{1-\delta_2}^1 \cdot^2 dxdz\right)^{1/2}$. Similarly define $\|\cdot\|_{\delta_1,z} = \left(\int_0^{\delta_1}(\cdot)^2 dz\right)^{1/2}$. If we choose $C = \frac{1}{\sqrt{2}}$ we find $\left|\int\int_0^{\delta_1} u_x u_z dzdx\right| \le \frac{\delta_1^2}{16}\|\nabla u\|^2$. We can perform an identical analysis at the top boundary layer, and set $\delta_1 = \delta_2$ to prove that

$$Q \ge \frac{1}{2}\|\nabla u\|^2 - Gr\frac{\delta_1^2}{8}\|\nabla u\|_{\delta_1}^2 - Gr\frac{\delta_1^2}{8}\|\nabla u\|_{1-\delta_2}^2 \tag{73}$$

$$\ge \left(\frac{1}{2} - Gr\frac{\delta_1^2}{8}\right)\|\nabla u\|^2 \tag{74}$$

This is positive if $\delta_1 \le 2Gr^{-1/2}$. This means that $\bar{u}_x(1) \ge 4Gr^{1/2}$. In terms of units $\bar{u}_x(h) = \bar{u}_x(1)\frac{\nu}{h} \ge 4\tau^{1/2}$. The friction coefficient $C_f = \frac{\tau}{\bar{u}(h)^2} \le \frac{1}{16}$.

2.4 Bounds in Three Dimensions

We can use much of the same algebra to derive bounds for the three dimensional case. The equations are the same except that there is a y component in the velocity field. As a result we have the same $\bar{u}(h) \ge 2U(1) - \frac{1}{Gr}\left\langle\left(\frac{dU}{dz}\right)^2\right\rangle$ as long as $Q \ge 0$. We pick the same background profile as in the two dimensional case, linear near each boundary and constant in the bulk.

We begin with the second term in Q and find that:

$$\left|\int\int\int_0^{\delta_1} u_x u_z dxdydz\right| \le \int\int\int_0^{\delta_1} z\|\frac{\partial u_x}{\partial z}\|_{\delta_1,z}\|\frac{\partial u_z}{\partial z}\|_{\delta_1,z} dxdydz \tag{75}$$

$$\le \frac{\delta_1^2}{2}\left(\frac{1}{2C}\int dx\int dy\|\frac{\partial u_x}{\partial z}\|_{\delta_1,z}^2 + \frac{C}{2}(1-a)\int dx\int dy\|\frac{\partial u_z}{\partial z}\|_{\delta_1,z}^2 + \frac{C}{2}a\int dx\int dy\|\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\|_{\delta_1,y}^2\right) \tag{76}$$

$$= \frac{\delta_1^2}{4}\left(\frac{1}{C}\|\frac{\partial u_x}{\partial z}\|_{\delta_1}^2 + C(1-a)\|\frac{\partial u_z}{\partial z}\|_{\delta_1}^2 + Ca\|\frac{\partial u_x}{\partial x}\|_{\delta_1}^2 + Ca\|\frac{\partial u_y}{\partial y}\|_{\delta_1}^2 + Ca\|\frac{\partial u_x}{\partial y}\|_{\delta_1}^2 + Ca\|\frac{\partial u_y}{\partial x}\|_{\delta_1}^2\right) \tag{77}$$

Choose $a = \frac{1}{2}$ and $C = \sqrt{2}$ to find that $\left|Gr\int\int\int_0^{\delta_1} u_x u_z dxdydz\right| \le \frac{\delta_1^2}{4\sqrt{2}}\|\nabla u\|_{\delta_1}^2$. We make the transformation $z \mapsto 1-z$ to study the upper boundary layer:

$$\left|\int dx\int dy\int_0^{\delta_2} u_x u_z dz\right| \le \int dx\int dy\int_0^{\delta_2} dz\frac{2}{\pi}\left(\int_{\delta_2}^1(\frac{\partial u_x}{\partial z})^2 dz'\right)^{1/2}\sqrt{z}\|\frac{\partial u_z}{\partial z}\|_{1-\delta_2,z} \tag{78}$$

$$\le \frac{3\delta_2^{3/2}}{2\pi}\left(\frac{1}{C}\|\frac{\partial u_x}{\partial z}\|^2 + \frac{C}{2}\|\frac{\partial u_z}{\partial z}\|_{\delta_2}^2 + \frac{C}{2}\|\frac{\partial u_x}{\partial x}\|_{\delta_2}^2 + \frac{C}{2}\|\frac{\partial u_y}{\partial y}\|_{\delta_2}^2\right) \tag{79}$$

Combining these two terms we find that:

$$|Q - \frac{1}{2}\|\nabla u\|^2| \le Gr\frac{\delta_1^2}{4\sqrt{2}}\|\nabla u\|_{\delta_1}^2 + Gr\frac{3\delta_2^{3/2}}{2\pi}\left(\frac{1}{C}\|\frac{\partial u_x}{\partial z}\|^2 + \frac{C}{2}\|\nabla u\|_{\delta_2}^2\right) \tag{80}$$

$$\le Gr\left(\left[\frac{\delta_1^2}{4\sqrt{2}} + \frac{3\delta_2^{3/2}}{2\pi}\frac{1}{C}\right]\|\frac{\partial u_x}{\partial z}\|^2 + \frac{\delta_1^2}{4\sqrt{2}}\|\nabla u\|^2\right) \tag{81}$$

$$= Gr\left(\left[\frac{\delta_1^2}{4\sqrt{2}} + \frac{3\sqrt{2}\delta_2^3}{2\pi^2\delta_1}\right]\|\frac{\partial u_x}{\partial z}\|^2 + \frac{\delta_1^2}{4\sqrt{2}}\|\nabla u\|^2\right) \tag{82}$$

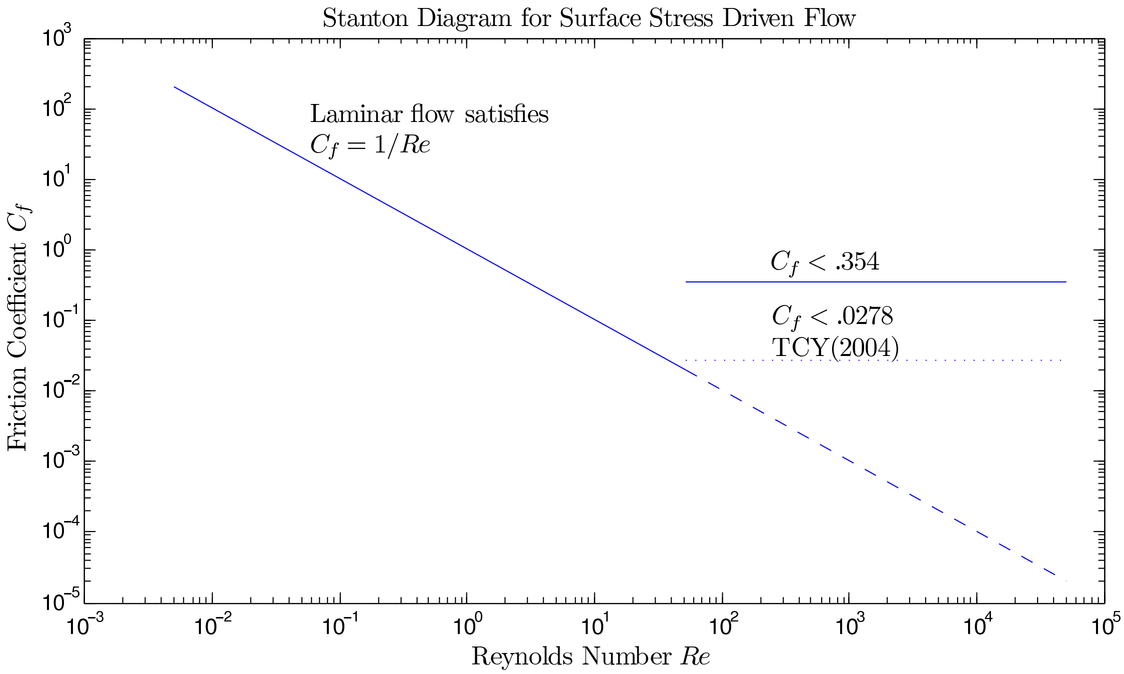


Figure 2: Stanton Diagram of the TCY bounds and the bounds proven here on C_f

In the last line we made the choice $C = \frac{\pi\delta_1^2}{3\sqrt{2}\delta_2^{3/2}}$. This shows that the ratio of the boundary layers $\frac{\delta_2}{\delta_1} \lesssim Gr^{-1/6}$. In order to produce a prefactor for the scaling of $\bar{u}(1)$ we take $\delta_2 = 0$. Then $\delta_1 = \sqrt{2\sqrt{2}}Gr^{-1/2}$ and $\bar{u}(h) \geq \sqrt{2\sqrt{2}}\tau^{1/2}$. The corresponding friction coefficient is $C_f \leq \frac{1}{2\sqrt{2}} = .354$. The bound proved by Tang, Caulfield, and Young[6] is $C_f < .0237$, which is stronger than this bound.

3 Infinite Prandtl Number Marangoni Convection

Consider the equations describing infinite prandtl number Marangoni convection:

$$\nabla p = \Delta u \tag{83}$$

$$\frac{\partial T}{\partial t} + u \cdot \nabla T = \Delta T \tag{84}$$

$$u|_{z=0} = 0 \tag{85}$$

$$T|_{z=0} = 0 \tag{86}$$

$$u_z|_{z=1} = 0 \tag{87}$$

$$\frac{\partial T}{\partial z}|_{z=1} = -1 \tag{88}$$

$$\frac{\partial u_x}{\partial z}|_{z=1} = -Ma\frac{\partial T}{\partial x}|_{z=1} \tag{89}$$

Take the divergence of the momentum equation:

$$\Delta p = \nabla \cdot \Delta u \tag{90}$$
$$= \Delta \nabla \cdot u \tag{91}$$
$$= 0 \tag{92}$$

If we take the Laplacian of the momentum equation we will find that u solves the biharmonic equation:

$$\Delta^2 u = \Delta \nabla p \tag{93}$$
$$= \nabla \Delta p \tag{94}$$
$$= 0 \tag{95}$$

We will be interested in the z component of the velocity for the background method. If we take the Fourier transform in the z direction it satisfies an ordinary differential equation:

$$\frac{d^4 u_z}{dz^4} - 2k^2 \frac{d^2 u_x}{dz^2} + k^4 u_x = 0 \tag{96}$$
$$u_z|_{z=0} = 0 \tag{97}$$
$$\frac{du_z}{dz}|_{z=0} = 0 \tag{98}$$
$$u_z|_{z=1} = 0 \tag{99}$$
$$\frac{d^2 u_z}{dz^2}|_{z=1} = k^2 Ma\theta(1) \tag{100}$$

This has an exact solution:

$$u_k = -Ma\theta_k(1) \frac{2\sinh(k)}{\frac{2\sinh(k)\cosh(k)}{k} - 2} \left(\sinh(kz) - kz\cosh(kz) + (k\coth(k) - 1)\, z \sinh(kz)\right) \tag{101}$$

Define $f_k(z) = \frac{u_k}{Ma\theta_k(1)}$.

We have plotted the modes in Figure 3. For large value of k the functions are concentrated near $z = 1$. For very large values of k the maximum value of f_k goes to zero. For small values of k the maximum of f_k is extremely small and the function is concentrated over the entire unit interval. For values of k near 3 and 4 the maximum is large and the concentration is also over a significant portion of the unit interval.

Set up the background method by making the substitution $T = \tau(z) + \theta$, where $\tau(0) = 0$ and $\frac{d\tau}{dz}(1) = -1$. Then the field θ satisfies the equation:

$$\frac{\partial \theta}{\partial t} + u_z \frac{d\tau}{dz} + u \cdot \nabla \theta = \Delta \theta + \frac{d^2 \tau}{dz^2} \tag{102}$$
$$\theta(z = 0) = 0 \tag{103}$$
$$\frac{\partial \theta}{\partial z}(1) = 0 \tag{104}$$

If we multiply the θ evolution equation by θ and integrate over space we get the expression:

$$\frac{1}{2} \frac{\partial \|\theta\|^2}{\partial t} = -\int dx \int dz \left(\theta u_z \tau' - \theta \tau''\right) - \|\nabla \theta\|^2 \tag{105}$$

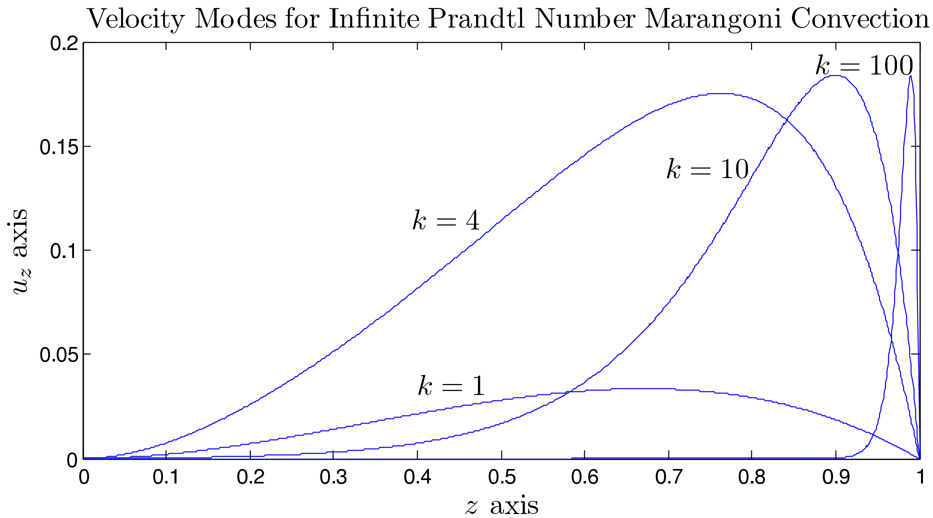


Figure 3: Plots of the functions f_k.

3.1 Energy Stability

Make the substitution $\tau = -z$ in the previous equation. The solution $\tau = -z$ will be stable when the quadratic form $Q = \|\nabla\theta\|^2 - \int u_z \theta dA$ is positive definite. Allow u_k to stand for the kth Fourier coefficient of u_z. Substitute the expression for the Fourier series of θ and u_z into Q:

$$Q = \sum_k \int_0^1 dz \left(-2\Re(\theta_k u_k) + (\frac{d\theta_k}{dz})^2 + k^2\theta_k^2 \right) dz \tag{106}$$

Then use the Cauchy-Schwarz inequality:

$$\begin{aligned} |\int_0^1 dz 2\Re(\theta_k u_k)| &\leq Ma\theta_k(1)\|f_k\|\|\theta_k\| & (107)\\ &\leq Ma\|\frac{d\theta_k}{dz}\|\|\|\theta_k\|\|\|f_k\| & (108)\\ &\leq \frac{Ma}{2}\|f_k\|\left(k\|\frac{d\theta_k}{dz}\|^2 + \frac{1}{k}\|\theta_k\|^2\right) & (109)\end{aligned}$$

If for each k the quantity $\frac{Ma}{2k}\|f_k\|$ is less than 1 the form Q will be positive definite. Since we know the f_k we can write this condition explicitly:

$$\left(\int_0^1 dz \left[\frac{2\sinh(k)}{\frac{2\sinh(k)\cosh(k)}{k} - 2}\left(\sinh(kz) - kz\cosh(kz) + (k\coth(k) - 1)\, z\sinh(kz)\right)\right]^2\right)^{1/2} \leq \frac{2k}{Ma} \tag{110}$$

This allows us to calculate the critical Marangoni number to be $Ma_c > 58.3$. The maximum occurs at $k = 2.4$. This estimate is slightly better than the estimate published Davis in

1969. Davis used variational methods and his bound is valid for all Prandtl numbers. The variational problem that he set up used natural boundary conditions to eliminate the terms that come from integrating by parts along the boundary. This weakening of the boundary condition is what makes it possible to improve the scaling using nonvariational methods.

3.2 Upper Bounds on the Nusselt Number

We define the Nusselt number to be the ratio of heat transport to heat transport due to conduction, so that it measures the strength of convection. Conduction requires a temperature gradient, so one would expect that when convection is strong the fixed heat flux is maintained with a minimal temperature gradient. The result is that the temperature at the top becomes low. Therefore we define the Nusselt number as $-\frac{1}{T(z=1)}$. If we multiply the time evolution equation for T by T and average we find that:

$$\frac{1}{Nu} = \|\nabla T\|^2 \tag{111}$$

Take the energy evolution equation for $\|\theta\|^2$ and time average it:

$$0 = -\int dx \int dz(\theta u_z \tau' - \theta\tau'') - \|\nabla\theta\|^2 \tag{112}$$

Now use the expression $\|\nabla\theta\|^2 = \|\nabla T\|^2 - \|\tau'\|^2 - 2\int dxdz\tau'\frac{\partial\theta}{\partial z}$ to write:

$$0 = -2\int dx \int dz(\theta u_z \tau' - \theta\tau'') - \|\nabla\theta\|^2 - \|\nabla T\|^2 + 2\int dxdz\tau'\frac{\partial\theta}{\partial z} + \|\tau'\|^2 \tag{113}$$

$$0 = -2\int dx \int dz(\theta u_z \tau') - \|\nabla T\|^2 - 2\bar{\theta}(1) - \|\nabla\theta\|^2 + \|\tau'\|^2 \tag{114}$$

$$0 = -2\int dx \int dz(\theta u_z) - 2\bar{T}(1) + 2\bar{\tau}(1) - \|\nabla\theta\|^2 + \|\tau'\|^2 + \|\nabla T\|^2 \tag{115}$$

$$-\frac{1}{Nu} = -2\int dx \int dz(\theta u_z \tau') - \|\nabla\theta\|^2 + 2\bar{\tau}(1) + \|\tau'\|^2 \tag{116}$$

If the functional $Q = 2\int dx \int dz(\theta u_z \tau') + \|\nabla\theta\|^2$ is always positive we can prove an upper bound on the Nusselt number:

$$\frac{1}{Nu} \geq -2\bar{\tau}(1) - \int dz\tau'^2 \tag{117}$$

We choose the derivative of the background profile τ to be equal to -1 in a layer of width δ_1 near the bottom and δ_2 near the top. We choose τ' to be constant in the bulk:

$$\tau(z) = -z \qquad 0 < z < \delta_1 \tag{118}$$

$$= -\delta_1 \qquad \delta_1 < z < 1-\delta_2 \tag{119}$$

$$= -\delta_1 - z + 1 - \delta_2 \qquad 1-\delta_2 < z < 1 \tag{120}$$

Then $\frac{1}{Nu} \geq \delta_1 + \delta_2$ as long as $\int dx \int dz(2\theta u_z) + \|\nabla\theta\|^2$ is positive. Write this in terms of the Fourier decomposition of θ and u_z:

$$0 \leq \sum_k \left[2\int_0^{\delta_1} dz\Re(\theta_k u_k) + 2\int_{1-\delta_2}^1 dz\Re(\theta_k u_k) + \|\frac{d\theta_k}{dz}\|^2 + k^2\|\theta_k\|^2\right] \tag{121}$$

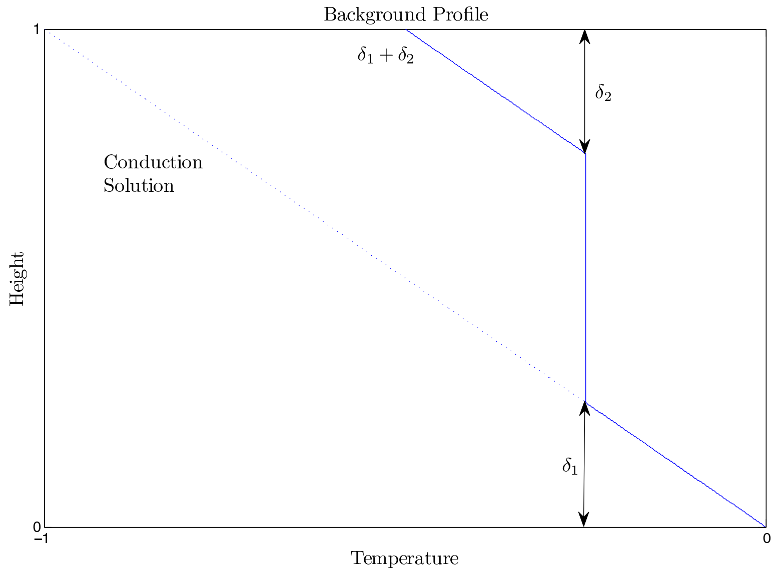


Figure 4: Background profile for Marangoni convection.

If for all k the quantity $2\int_0^{\delta_1} dz\Re(\theta_k u_k) + 2\int_{1-\delta_2}^{1} dz\Re(\theta_k u_k) + \|\frac{d\theta_k}{dz}\|^2 + k^2\|\theta_k\|^2 > 0$ the form Q will be positive definite.

We begin the analysis with the lower boundary layer. We start by replacing u_k with $Ma\theta_k(1)f_k$. Then we use the fundamental theorem of calculus:

$$|\int_0^{\delta_1} f_k\theta_k(1)\theta_k dz| \leq \int_0^{\delta_1} |f_k\theta_k|dz\|\frac{d\theta_k}{dz}\| \tag{122}$$

$$\leq \int_0^{\delta_1} \sqrt{z}|f_k|\|\frac{d\theta_k}{dz}\|^2 \tag{123}$$

Let $F(z) = sup_k f(z)$. Then we need to evaluate $\int_0^{\delta_1} \sqrt{z}F(z)dz$ to determine the scaling. As z goes to zero each f_k scales as $c_k z^2$ so there is reason to believe that F might have the same behavoir. We verified numerically that for extremely small z the value of k at which the maximum over f_k is realized has a lower bound, implying that there is some c such that $F(z) < cz^2$. We show this in Figure 3.2. This means we can choose $\delta_1^{7/2} = O(Ma^{-1})$, or that $Nu < O(Ma^{2/7})$. We can calculate the prefactor numerically. We set $\delta_2 = 0$. We find that $Nu < .84Ma^{2/7}$.

Boeck and Thess have used numerical methods to find solutions of the infinite Prandtl number Marangoni problem. They analyzed the scaling of the Nusselt number of their numerical solutions with respect to the Marangoni number and found that $Nu = .446Ma^{.238}$. They also theorized that $Nu \sim Ma^{2/9}$ [1]. We compare their data to our bound in Figure 3.2.

3.3 Heat Transport with finite Prandtl Number

We consider the problem of Marangoni Convection in a system with finite Prandt number. The system is described by the equations:

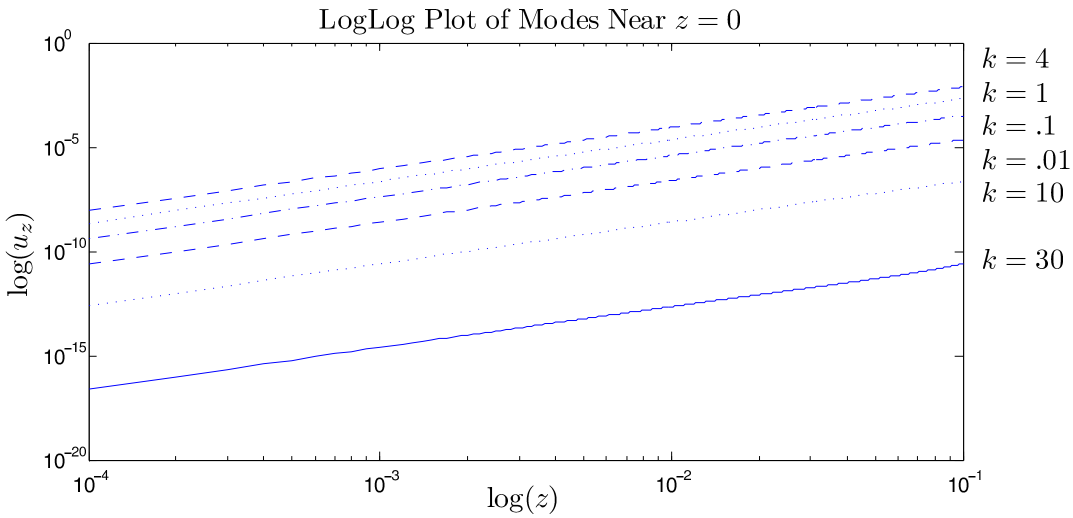


Figure 5: Scaling of the functions f_k for different values of k near $z = 0$.

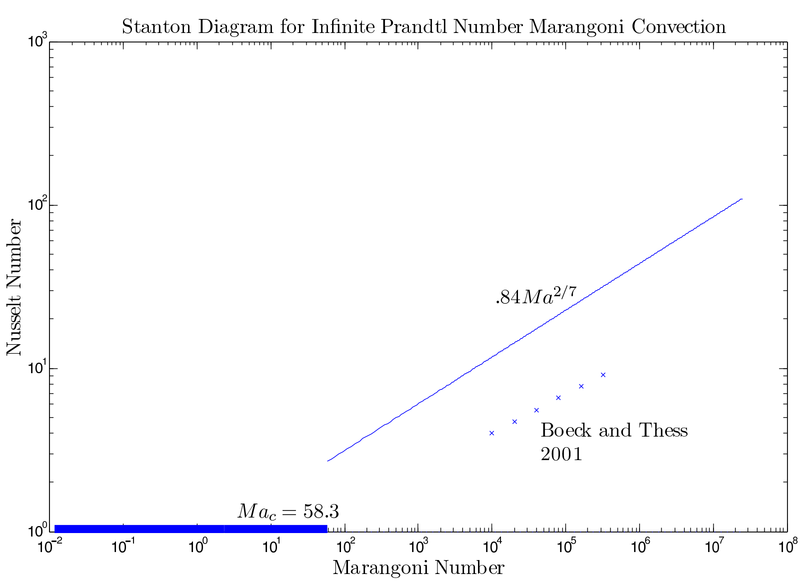


Figure 6: Plot of Nusselt and Marangoni number of numerical data compared to the rigorous upper bound and the critical Marangoni number

$$P^{-1}(\frac{\partial u}{\partial t} + u \cdot \nabla u) = -\nabla p + \Delta u \tag{124}$$

$$\nabla \cdot u = 0 \tag{125}$$

$$\frac{\partial T}{\partial t} + u \cdot \nabla T = \Delta T \tag{126}$$

$$u(z=0) = \vec{0} \tag{127}$$

$$T(z=0) = 1 \tag{128}$$

$$u_3(z=1) = 0 \tag{129}$$

$$\frac{\partial T}{\partial z}(z=1) = -1 \tag{130}$$

$$\frac{\partial u_1}{\partial z} = -Ma\frac{\partial T}{\partial x}(z=1) \tag{131}$$

$$\frac{\partial \theta}{\partial z}(z=1) = 0 \tag{132}$$

The Nusselt number of this flow is defined as $Nu = -\frac{1}{T(1)}$. Multiply the equation for the temperature by T and average it over the volume. All the terms on the left hand side vanish, the first term because the norm of temperature is bounded and the advection term because there is no flux across the upper and lower boundaries. Therefore we find that:

$$0 = \int_V T\Delta T dV \tag{133}$$

$$= -\|\nabla T\|^2 + \int_{\Gamma V} T\frac{\partial T}{\partial z}dA \tag{134}$$

$$= -\int_{z=1} TdA - \|\nabla T\|^2 \tag{135}$$

This means that $Nu = \frac{1}{\|\nabla T\|^2}$. Introduce a background profile $\tau(z)$. We choose τ to satisfy a homogeneous Dirichlet condition at $z=0$ and the Neumann condition $\frac{\partial \tau}{\partial z} = -1$ at $z=1$. Then we can decompose the temperature $T = \tau(z) + \theta(x,z,t)$ where $\theta = 0$ at $z=0$ and $\frac{\partial \theta}{\partial z} = 0$ at $z=1$. For each τ there is an equation for θ:

$$\frac{\partial \theta}{\partial t} + u \cdot \nabla \theta + u_z\frac{\partial \tau}{\partial z} = \Delta \theta + \frac{\partial^2 \tau}{\partial z^2} \tag{136}$$

Multiply this equation by θ and take the average over space and time. Again the first two terms on the left hand side vanish:

$$0 = -\|\nabla \theta\|^2 - \bar{\theta}(1) - \langle \theta u_z \tau' \rangle - \langle \frac{\partial \theta}{\partial z} \rangle \tag{137}$$

Now substitute τ and θ for T into $\|\nabla T\|^2 = \frac{1}{Nu}$:

$$\frac{1}{Nu} = \|\nabla \theta\|^2 + \|\nabla \tau\|^2 + 2\int \nabla \theta \cdot \nabla \tau dV \tag{138}$$

$$= -\|\nabla \theta\|^2 - 2\bar{\theta}(1) - 2\langle \theta u_z \tau' \rangle + \int \tau'^2 dv \tag{139}$$

In the last line we added two times the identity in the previous equation. In order to incorporate the Marangoni condition we need the momentum equation. We take the dot product of the momentum equation with u and calculate the space time average:

$$0 = -\|\nabla u\|^2 + \int_{z=1} u_x \frac{\partial u_x}{\partial z} dA \tag{140}$$

$$= -\|\nabla u\|^2 - Ma \int_{z=1} u_x \frac{\partial \theta}{\partial x} dA \tag{141}$$

$$= -\|\nabla u\|^2 - Ma \int_V \frac{\partial}{\partial z}(u_x \frac{\partial \theta}{\partial x}) dV \tag{142}$$

Consider linear combinations of this expression and the equation for the Nusselt number:

$$\frac{1}{Nu} = -\|\nabla\theta\|^2 - 2\bar{\theta}(1) - 2\langle\theta u_z \tau'\rangle + \int_V \tau'^2 dv + C(\|\nabla u\|^2 + Ma\int_V \frac{\partial}{\partial z}(u_x\frac{\partial\theta}{\partial x})dV) \tag{143}$$

$$= -\|\nabla\theta\|^2 - 2(\bar{T}(1) - \bar{\tau}(1)) - 2\langle\theta u_z \tau'\rangle + \int_V \tau'^2 dv + C(\|\nabla u\|^2 + Ma\int_V \frac{\partial}{\partial z}(u_x\frac{\partial\theta}{\partial x})dV) \tag{144}$$

$$= -2\bar{\tau}(1) - \langle\tau'^2\rangle + Q(\theta,\tau,u) \tag{145}$$

In the second to last line we used the fact that $\theta = T - \tau$. In the last line we used the fact that $Nu = -\frac{1}{\bar{T}(1)}$ and defined the quadratic form Q:

$$Q(\theta,\tau,u) = \|\nabla\theta\|^2 + 2\langle\theta u_z \tau'\rangle + C(\|\nabla u\|^2 + Ma\int \frac{\partial}{\partial z}(u_x\frac{\partial\theta}{\partial x})) \tag{146}$$

Suppose that $Q > 0$ for all u, θ, for some given choice of τ. Then $\frac{1}{Nu} < -2\bar{\tau}(1) - \langle\tau'^2\rangle$. The bound on the Nusselt number will only depend on the choice of τ. Therefore the task will be to choose a τ that maximizes the bound on $\frac{1}{Nu}$ subject to the constraint that Q is positive. The background τ only affects Q through the term $\langle\theta u_z \tau\rangle$, which can be negative. Therefore it would make sense to make this term as small as possible. Define τ such that τ' is zero everywhere except near the upper and lower boundaries.

$$\tau = -z \qquad z < \delta_1 \tag{147}$$

$$\tau = -\delta_1 \qquad \delta_1 < z < 1-\delta_2 \tag{148}$$

$$\tau = -\delta_1 - \delta_2 + 1 - z \qquad 1-\delta_2 < z < 1 \tag{149}$$

With this choice of τ the expression $\langle\theta u_z \tau'\rangle$ reduces to the inner product of u_z and θ over regions of width δ_1 and δ_2 near the boundaries. The τ also satisfies the boundary conditions. We treat the top and bottom layer separately, starting with the bottom layer:

$$\int_0^{\delta_1} \theta u_z dV \le \int_0^{\delta_1} \left(\int_0^z |\frac{\partial}{\partial z'}\theta dz'|\right)\left(\int_0^z |\frac{\partial}{\partial z''}u_z dz''|\right) dxdz \tag{150}$$

$$\le \int_0^{\delta_1} z\|\frac{\partial\theta}{\partial z}\|\|\frac{\partial u_z}{\partial z}\| dxdz \tag{151}$$

$$= \frac{\delta_1^2}{2}\|\frac{\partial\theta}{\partial z}\|\|\frac{\partial u_z}{\partial z}\| \tag{152}$$

It is more difficult to deal with the upper layer because θ is not zero on the upper boundary. As a result we have to use the Poincare inequality instead of the method we just used on the lower layer. We can choose $\delta_2 = 0$ and not effect the scaling.

The term involving the Marangoni number must also be controlled by the norms of the derivatives of u and θ.

$$Ma \int \frac{\partial}{\partial z}\left(u_x \frac{\partial \theta}{\partial x}\right) = Ma \int \left(\frac{\partial u_x}{\partial z}\frac{\partial \theta}{\partial x} - \frac{\partial u_x}{\partial x}\frac{\partial \theta}{\partial z}\right) \tag{153}$$

$$\leq \|\frac{\partial u_x}{\partial z}\|\|\frac{\partial \theta}{\partial x}\| + \|\frac{\partial u_z}{\partial z}\|\|\frac{\partial \theta}{\partial z}\| \tag{154}$$

Here we have used the fact that u is incompressible. Now we have bound on Q in terms of first derivatives of u and θ. We are only interested in the scaling and do not make an effort here to derive an optimal bound. If we take $C = 1$ we find that Q is positive when the following equation for λ has no negative solutions:

$$(1-\lambda)^4 - (\frac{\delta^4}{4} + \frac{Ma\delta^2}{2})(1-\lambda)^2 = 0 \tag{155}$$

This is satisfied as long as $\frac{\delta^4}{4} + \frac{Ma\delta^2}{2} < 1$. This suggests that for large Ma $\delta \lesssim Ma^{-1/2}$. This suggests that $Nu \lesssim Ma^{1/2}$.

The scaling $Nu \lesssim Ma^{1/2}$ is considerably weaker than the infinite Prandtl number case. On the other hand, simulations by Boeck and Thess do not indicate that the actual scaling is weaker for the finite Prandtl number case. The reason that this scaling is so much worse is because we were unable to take advantage of the relationship between the temperature and velocity fields. It was possible to achieve the 2/7th bound because we knew what the Fourier modes of the velocity field were. However the relationship is much more complicated in the finite Prandtl number case and we were unable to use it. In general, improvements of the bounds produced by the background method in Raleigh-Benard convection were achieved by using the relationship [2], so it seems reasonable to expect that this bound could be improved by such considerations.

4 Conclusion

We presented rigorous upper bounds for the friction coefficient in stress driven shear flow and also derived the critical Grashoff number for the stability of the laminar flow solution. Tang, Caulfield, and Young first used the background method to analyze this problem. We were able to prove an analytic bound on the friction coefficient using the full stress boundary condition. We improved the lower bound on the critical Marangoni number for nonlinear stability in infinite dimensional Prandtl number Marangoni convection. We also proved that $Nu < .84Ma^{2/7}$ for the same problem. In finite Prandtl number Marangoni convection we were unable to do any better than $Nu \sim Ma^{1/2}$.

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