Energy and dissipation in MHD systems

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1 Introduction

In this work, Hartmann flow is studied in the framework of imcompressible magnetohydrodynamics. In the first sections, we present simple examples of magnetic problem showing the role of the vaccum or an insulator on the energy and on the dissipation rate of a conductor. Then, we derive exact laminar solution and explicit expression for the energy stability of the Hartmann flow in contact with an insulator. We show that taking into account realistic boundary conditions for the magnetic field yields technical difficulties in the derivation of exact bounds on the dissipation rate using the background method.

2 Dissipation in a cylindrical conductor-1

In this section we study a simple situation of free decay of a magnetic field. We consider a wire of metal with conductivity σ and permeability μ. The wire is a cylinder infinite in the axial direction z and have a finite radius a. The cylinder is surrounded by vaccum(see figure **??**). We suppose that there is a current $\mathbf{J}$ only at $t = 0$ and we study the evolution of the magnetic field created by the initial current.

In this first simple problem, the intial current is a toroidal current depending only on the radial direction:

$$\mathbf{J}(\mathbf{r}) = J_\theta(r)\mathbf{e}_\theta \tag{1}$$

By symmetry and use of Ampere's law, we know that the corresponding initial magnetic field created is $B_z(t, r)\mathbf{e}_z$ inside the wire and is zero outside (for $r > a$). Thus, the governing equation are:

$$\frac{\partial \mathbf{B}}{\partial t} = \eta \Delta \mathbf{B} \tag{2}$$

which is in cylindrical corrdinates for the magnetic field inside the conducting domain is :

$$\frac{\partial \mathbf{B_z}}{\partial t} = \eta[-\frac{\partial^2 B_z}{\partial r^2} - \frac{1}{r}\frac{\partial B_z}{\partial r}] \tag{3}$$

where we have introduced the magnetic diffusivity $\eta = 1/(\sigma\mu)$. The magnetic field is simply $\nabla \times \mathbf{B} = 0$ for $r > a$. In this diffusive situation, the magnetic field can only decay and we suppose that $B_z(t, r) = b(r)e^{-\alpha t}$. The equation (3) then reduce to :

$$\frac{\partial^2 b}{\partial r^2} + \frac{1}{r}\frac{\partial b}{\partial r} + \frac{\alpha}{\eta} b = 0 \tag{4}$$

Using the change of variables $\rho = r\sqrt{\alpha/\eta}$, we obtain:

$$\frac{\partial^2 b(\rho)}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial b(\rho)}{\partial \rho} + b(\rho) = 0 \tag{5}$$

We recognize the bessel equation for $m = 0$. A solution of the equation is thus:

$$b = J_0(\sqrt{\frac{\alpha}{\eta}} r) \tag{6}$$

The boundary conditions for the magnetic field leads to an expression for the zeros of the bessel function:

$$B_z(a) = 0 \Longrightarrow J_0(\sqrt{\frac{\alpha}{\eta}} a) = 0 \tag{7}$$

The n^{th} zero of the bessel function $J_0(\rho)$ is given by $\lambda_n = \sqrt{\frac{\alpha_n}{\eta}} a$. We see that the boundary conditions yields here a discretization of the possible decay rates for the magnetic field. This is not surprising since these boundary conditions result in a confinement of the field in a finite domain. We can then expand the fields in Fourier-Bessel series for $r < a$:

$$b(r,t) = \sum_{n=0}^{\infty} A_n J_0(\lambda_n r/a) e^{-t\eta\lambda_n^2/a^2} \tag{8}$$

And simply $b(r,t) = 0$ for $r > a$. Because Bessel's equation is Hermitian, the solutions must satisfy the following orthogonality relationship :

$$\int_0^1 J_\nu(x\lambda_m^\nu) J_\nu(x\lambda_n^\nu) x dx = \frac{\delta_{mn}}{2}[J_{\nu+1}(\lambda_n^\nu)]^2 \tag{9}$$

In order to express the coefficient of the magnetic field we evaluate the integral :

$$I = \int_0^1 r dr J_0(\lambda_n r) B^o(r) \tag{10}$$

Where $B^o(r)$ represent the magnetic field at $t = 0$. Expansion of $B^o(r)$ gives:

$$I = \int_0^1 r dr J_0(\lambda_n r) \sum_{m=1}^{\infty} A_m J_0(\lambda_m r) \tag{11}$$

$$I = \sum_{m=1}^{\infty} A_m \int_0^1 r dr J_0(\lambda_n r) J_0(\lambda_m r) \tag{12}$$

$$I = \sum_{m=1}^{\infty} \frac{A_m}{2} \delta_{mn} [J_1(\lambda_m)]^2 \tag{13}$$

$$I = \frac{A_n}{2}[J_1(\lambda_n)]^2 \tag{14}$$

We then get an expression for the coefficient of the magnetic field:

$$A_n = \frac{2}{[J_1(\lambda_m)]^2}\int_0^1 rdrJ_0(\lambda_n r)B^o(r) \tag{15}$$

The decomposition of the magnetic field on the basis of Bessel functions with well defined coefficients provide a natural way for evaluating the magnetic energy in the volume. Indeed the Parseval theorem applied to Bessel-Fourier series gives:

$$\epsilon = \int dV[\mathbf{B}(\mathbf{r},\mathbf{t})]^2 = \sum_{n=0}^{\infty}|A_n e^{-t\eta\lambda_n^2/a^2}|^2 = \sum_{n=0}^{\infty}|A_n|^2 e^{-2t\eta\lambda_n^2/a^2} \tag{16}$$

This yields to a negative time variation of the energy:

$$\dot{\epsilon} = -2\eta\sum_{n=0}^{\infty}\lambda_n^2|A_n|^2 e^{-2t\eta\lambda_n^2/a^2} \tag{17}$$

Finally, we use the fact that all the zeros of J_0 are greater or equal to the first zero λ_1 of the Bessel function:

$$\dot{\epsilon} = -2\eta\lambda_1^2/a^2\sum_{n=0}^{\infty}\frac{\lambda_n^2}{\lambda_1^2}|A_n|^2 e^{-2t\eta\lambda_n^2/a^2} \leq -2\eta\lambda_1^2\epsilon \tag{18}$$

The gronwall inequality leads finally to a minimal rate of variation for the energy:

$$\epsilon(t) \leq \epsilon(0)e^{\frac{2\eta\lambda_0^2}{a^2}t} \tag{19}$$

3 Dissipation in a cylindrical conductor-2

In the previous problem, the initial current lead to very simple boundary conditions preventing the magnetic field to come out from the cylinder. It is thus interesting to study a similar problem but with different boundary conditions.

In this section, we consider a problem with the same geometrical configuration than in the previous section but with a different initial current. At $t = 0$ (and $t = 0$ only), we impose an axial current depending only on r:

$$\mathbf{J}(\mathbf{r}) = J_z(r)\mathbf{e}_z \tag{20}$$

Obviously, the magnetic field created by this current will be of the form $B_\theta(t,r)\mathbf{e}_\theta$. Outside the conductor, the magnetic field is now non-zero and satisfy the equation $\nabla \times B = 0$, yielding an harmonic field going to zero at infinity. Inside the conductor the field obey the diffusion equation:

$$\frac{\partial B_\theta}{\partial t} = \eta[-\frac{\partial^2 B_\theta}{\partial r^2} - \frac{1}{r}\frac{\partial B_\theta}{\partial r} + \frac{B_\theta}{r^2}] \tag{21}$$

As before, the decaying magnetic field is $B_\theta(t,r) = b(r)e^{-\alpha t}$. By using the change of variable $\rho = r\sqrt{\alpha/\eta}$, the equation (21) become:

$$\frac{\partial^2 b(\rho)}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial b(\rho)}{\partial \rho} + b(\rho).(1 - \frac{1}{\rho^2}) = 0 \tag{22}$$

This is the bessel equation for degree $m = 1$. A solution of the equation is thus:

$$b = J_1(\sqrt{\frac{\alpha}{\eta}}r) \tag{23}$$

In this case, the boundary conditions do not provide sufficient constraint on the magnetic field and thus do not lead to vanishing conditions for Bessel functions. This means that the system can not be represented by a Fourier-Bessel serie. In particular, the decay rates α can not be discretized, which seems to be in agreement with the fact that the energy is infinite outside the cylinder.
Using Hankel transform, we can however expand the inner magnetic field as follow :

$$b_i(r,t) = \int_0^\infty d\alpha c(\alpha) J_1(\sqrt{\frac{\alpha}{\eta}}r)e^{-\alpha t} \tag{24}$$

Using the transform $\beta = \sqrt{\alpha/\eta}$, the equation (24) becomes:

$$b_i(r,t) = 2\eta \int_0^\infty \beta d\beta C(\beta) J_1(\beta r)e^{-\eta\beta^2 t} \tag{25}$$

defined only for $r < a$. Outside the conductor, the magnetic field is simply harmonic and thus have the form $b_o(r,t) = A(t)/r$. The continuity of the tangential component of $H = B/\mu$ across the boundary yields an expression for the outer field:

$$b_o(r,t) = 2\eta \frac{\mu_0 a}{\mu r} \int_0^\infty \beta d\beta C(\beta) J_1(\beta a)e^{-\eta\beta^2 t} \tag{26}$$

We will now suppose that it is possible to find some magnetic field b_T defined on all the domain$(0 < r < \infty)$ and represented on the basis of Bessel function:

$$b_T(r,t) = 2\eta \int_0^\infty \beta d\beta A(\beta) J_1(\beta r)e^{-\eta\beta^2 t} \tag{27}$$

In order to find an expression for the coefficent $A(\beta)$, we will evaluate the integral:

$$Y = \int_0^\infty r dr J_1(\beta r) b_T(r,0) \tag{28}$$

There is two way to evaluate Y. By considering the expansion of B_T :

$$Y = \int_0^\infty r dr J_1(\beta r) 2\eta \int_0^\infty \beta' d\beta' A(\beta') J_1(\beta' r)e^{-\eta\beta'^2 t} \tag{29}$$

Using the orthogonality condition :

$$\int_0^\infty r dr J_1(\beta r) J_1(\beta' r) = \frac{\delta(\beta - \beta')}{\beta} \tag{30}$$

the equation (29) become:

$$Y = 2\eta A(\beta) \tag{31}$$

We can also evaluate Y by separating the domain of integration:

$$Y = \int_0^a r dr J_1(\beta r) b_i(r,0) + \int_a^\infty r dr J_1(\beta r) b_o(r,0) \tag{32}$$

$b_o(r,0)$ is related to $b_i(r,0)$ and we get:

$$Y = \int_0^a r dr J_1(\beta r) b_i(r,0) + b_i(a,0)\frac{\mu_0 a}{\mu}\int_a^\infty dr J_1(\beta r) \tag{33}$$

By grouping equations (31) and (33) we finally get an expression for the coefficient $A(\beta)$ of the magnetic field:

$$A(\beta) = \frac{1}{2\eta}\int_0^a r dr J_1(\beta r) b_i(r,0) + b_i(a,0)\frac{\mu_0 a}{2\eta\mu}\int_a^\infty dr J_1(\beta r) \tag{34}$$

By comparison with the first problem where decay modes were quantified, it can be interesting to investigate the evolution of the energy of the system.The energy in the infinite volume is given by:

$$\epsilon = \int_0^\infty r dr b_T^2 = \int_0^\infty \beta d\beta |A(\beta)e^{-\eta\beta^2 t}|^2 \tag{35}$$

We separate this integral in two parts,using a small parameter w:

$$\epsilon = \int_0^w \beta d\beta |A(\beta)e^{-\eta\beta^2 t}|^2 + \int_w^\infty \beta d\beta |A(\beta)e^{-\eta\beta^2 t}|^2 \tag{36}$$

In the first part of the integral, $A(\beta)$ can be expressed by its Taylor expansion $A(\beta) = K\beta$ since $A(0) = 0$. In the second part, the asymptotic limit of long time t allow us to neglect this term to zero. The energy is then given by:

$$\epsilon \sim \int_0^\infty \beta^3 d\beta e^{-2\eta\beta^2 t} \tag{37}$$

By integration by parts we get $\epsilon \sim t^{-3/2}$. We can easily derive this equation and we finally get the result :

$$\frac{\dot{\epsilon}}{\epsilon} \sim \frac{\mu}{t} \tag{38}$$

In the previous problem, the magnetic field was restricted to a finite radius, yielding a quantification of the decay modes. As a consequence, the energy was bounded by the exponential decay of the less damped mode. In this new problem, the magnetic field is no longer restricted to a finite volume but goes to infinity. Moreover, the total energy is infinite outside the conductor. We see here that it is thus impossible to get some exponential decay for the magnetic energy. In this situation, the system is damping the energy with a much slower rate. This is probably reminiscent from the infinite amount of energy to dissipate coming from outside and collapsing on the conductor.

4 Decay problem in spherical geometry

We have seen how the magnetic field is decaying in a cylindrical conductor under the action of ohmic diffusion. In some case, it is impossible to observe exponential bound of the magnetic energy. In this section we study the problem of free decay modes in a spherical configuration.

Let's consider a spherical conductor of permeability μ and conductivity σ surrounded by infinite vaccum. In such a geometry, it is in general useful to decompose the soleloidal fields into poloidal part P and toroidal part T. The magnetic field is given by :

$$\mathbf{B} = \nabla \times \nabla \times (\mathbf{r}P) + \nabla \times (\mathbf{r}T) \tag{39}$$

The governing equations for the toroidal part are:

$$\frac{\partial T}{\partial t} = \eta \Delta T \, r < a \tag{40}$$

$$T = 0 \, r > a \tag{41}$$

Similarly we get two equations for the poloidal part:

$$\frac{\partial P}{\partial t} = \eta \Delta P \, r < a \tag{42}$$

$$\Delta P = 0 \, r > a \tag{43}$$

This decomposition in toroidal and poloidal part is interesting because P and T may always be decomposed on the basis of the spherical harmonics, which are the eigenfunction of the angular part of the spherical laplace operator.The problem for T and P are totally independant and can be sutdied separatly. I will present all the work as axisymmetric problem for simplicity but the results are totally similar for non-axisymmetic problem.

4.1 Toroidal decay problem

The scalar T can be decomposed as follow:

$$T = \sum_l \hat{T}^l Y_l(\theta, \phi) \tag{44}$$

Using this decomposition the equation (40) become:

$$\frac{\partial \hat{T}^l}{\partial t} = \eta \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \hat{T}^l}{\partial r} \right) - \frac{l(l+1)}{r^2} \hat{T}^l \right) \tag{45}$$

We know that it is a decaying problem so we can suppose $\hat{T}^l(r,t) = T^l(r) e^{-\alpha t}$. We then get the equation:

$$\frac{\partial^2 T^l}{\partial r^2} + \frac{2}{r} \frac{\partial T^l}{\partial r} + \left(\frac{\alpha}{\eta} + \frac{l(l+1)}{r^2} \right) T^l = 0 \tag{46}$$

This equation is the spherical bessel equation and a solution to this equation is :

$$T_\alpha^l = r^{-1/2} J_{l+\frac{1}{2}} \left(\sqrt{\frac{\alpha}{\eta}} r \right) \tag{47}$$

Boundary conditions for toroidal magnetic field are:

$$T^l(a) = 0 \Longrightarrow J_{l+\frac{1}{2}} \left(\sqrt{\frac{\alpha_n}{\eta}} a \right) = 0 \tag{48}$$

As for the problem 1, the vanishing of T at the boundary yield a determination of the zeros $\lambda_n = \sqrt{\frac{\alpha_n}{\eta}} a$ of the Bessel function. We can then expand T^l as a Fourier-Bessel serie:

$$T^l = \sum_{n=0}^{\infty} A_n r^{-1/2} e^{-t\eta\lambda_n^2/a^2} J_{l+\frac{1}{2}}(\lambda_n r) \tag{49}$$

The total representation of the toroidal magnetic field inside the conductor is thus:

$$T_i(r,t) = \sum_l Y_l(\theta,\phi) \sum_{n=0}^{\infty} A_n^l r^{-1/2} e^{-t\eta(\lambda_n^l)^2/a^2} J_{l+\frac{1}{2}}(\lambda_n^l r) \tag{50}$$

The field is now expanded in term of Bessel function but also of spherical harmonics. In addition of the orthogonality for bessel function (9) we have to consider the orthogonality of spherical harmonic:

$$\int d\Omega Y_l^m(\theta,\phi) Y_{l'}^{m'}(\theta,\phi) = \delta_{ll'}\delta_{mm'} \tag{51}$$

We can now use boundary conditions to compute the coefficient A_n^l. Because the toroidal field is vanishing on the boundary, it leads to consider integral restricted to the conducting domain. We evaluate thus the integral :

$$I = \int d\Omega \int_0^1 r^2 dr T_i(r,0) r^{-1/2} J_{l+\frac{1}{2}}(\lambda_n^l r) Y_l(\theta,\phi) \tag{52}$$

We expanding $T_i(r,0)$ and using orthogonality, I becomes:

$$I = \int d\Omega \int_0^1 r^2 dr \sum_{l'} \sum_{n'} A_{n'}^{l'} (r^{-1/2})^2 J_{l+\frac{1}{2}}(\lambda_n^l r) J_{l'+\frac{1}{2}}(\lambda_{n'}^{l'} r) Y_l(\theta,\phi) Y_l(\theta',\phi) \tag{53}$$

$$I = \sum_{l'} \int d\Omega Y_l(\theta,\phi) Y_l(\theta',\phi) \sum_{n'} A_{n'}^{l'} \int_0^1 r dr J_{l+\frac{1}{2}}(\lambda_n^l r) J_{l'+\frac{1}{2}}(\lambda_{n'}^{l'} r) \tag{54}$$

$$I = A_n^l [J_{l+\frac{3}{2}}(\lambda_n^l)]^2 \tag{55}$$

By changing bounds of the integral over r, we finally obtain an expression for A_n^l :

$$A_n^l = \frac{\sqrt{a\eta}}{a^3 [J_{l+\frac{3}{2}}(\lambda_n^l)]^2} \int dV_{cond} r^{-1/2} J_{l+\frac{1}{2}}(\lambda_n^l r/a) \tag{56}$$

We see here that the boundary conditions for the toroidal field lead to quantification of the modes and, as in the section 2, we expect that the energy will be dominated by the decay rate of the modes. Using orthogonality of Bessel fuction and spherical harmonic we get an expression for the evolution of the energy:

$$\epsilon_T = \sum_l \sum_n \frac{|A_n^l|^2}{2} e^{-2\eta(\lambda_n^l)^2 t/a^2} [J_{l+\frac{3}{2}}(\lambda_n^l)]^2 \tag{57}$$

By derivating this energy with respect to t and using Gronwall inequality like in section 2, we get an exponential bounds for the energy related to the less damped mode:

$$\epsilon_T \leq \epsilon_T(0) e^{-2t\eta(\lambda_1^1)^2/a^2} \tag{58}$$

4.2 Poloidal decay problem

Let's do the same thing for the poloidal part of the magnetic field. The SPherical harmonic decompostion is still possible:

$$P = \sum_l \hat{P}^l Y_l(\theta, \phi) \tag{59}$$

Inside the conductor, the equation is the same than for the toroidal part and we get the solution:

$$P_\gamma^l = r^{-1/2} J_{l+\frac{1}{2}}(\sqrt{\frac{\gamma}{\eta}} r) \tag{60}$$

Where γ denotes now the decay rate of poloidal component P_γ^l. The boundary conditions are however different in the case of the poloidal field. According to the equation (43) the scalar P is harmonic outside the conductor and $P^l(r,t) = c(t) r^{-(l+1)}$.The radial derivative of this field at the outer boundary is then related to P by:

$$\frac{\partial P_\gamma^l}{\partial r} + \frac{(l+1)}{a} P = 0 \tag{61}$$

By plugging expression (60) into equation (61) and using recurrence relation for the Bessel functions, we obtain:

$$J_{l-\frac{1}{2}}(\sqrt{\frac{\gamma}{\eta}} a) = 0 \tag{62}$$

We see here that the boundary conditions discretize the possible decay rate of the field as for toroidal part but involve now zero of $J_{l-\frac{1}{2}}$. This is easy to relate it to the zeros λ_n^l of $J_{l-\frac{1}{2}}$ and the decay rates of the modes are given by:

$$\gamma_n^l = \eta \frac{(\lambda_n^{l-1})^2}{a^2} \tag{63}$$

We thus have the following expansion for P inside the conductor:

$$P_i(r,t) = \sum_l Y_l(\theta, \phi) \sum_{n=0}^{\infty} B_n^l r^{-1/2} e^{-t\eta(\lambda_n^{l-1})^2/a^2} J_{l+\frac{1}{2}}(\lambda_n^{l-1} r) \tag{64}$$

Outside the conductor, the field is expanding only on the spherical harmonics, and is totally determined by the value of the internal field at the boundary:

$$P_o(r,t) = \sum_l c^l(t) Y_l(\theta,\phi) r^{-(l+1)} \tag{65}$$

Where we use the boundary conditions to calculate the coefficent $c^l(t)$:

$$c^l(t) = \sum_{n=0}^{\infty} B_n^l a^{l+1/2} e^{-t\eta(\lambda_n^{l-1})^2/a^2} J_{l+\frac{1}{2}}(\lambda_n^{l-1} a) \tag{66}$$

The determination of the coefficient is identical as for the toroidal field and give:

$$B_n^l = \frac{\sqrt{a\eta}}{a^3[J_{l+\frac{3}{2}}(\lambda_n^{l-1})]^2} \int dV_{cond} r^{-1/2} J_{l+\frac{1}{2}}(\lambda_n^{l-1} r/a) \tag{67}$$

Here again, the rate of decay of the magnetic energy will be governed by the decay rate of the less damped mode. However, because of the shift in the index l, the first eigenvalue is now given by $\lambda_1^0 = pi$ and we get the following bounds for the rate of decay of the energy:

$$\epsilon_P \leq \epsilon_P(0) e^{-\frac{2\eta\pi^2}{a^2}t} \tag{68}$$

Although the poloidal magnetic field is not constrained to a finite volume like the toroidal one, we see that it can also be represented as a discrete sum of magnetic modes, with quantification of the possible decay rate, yielding an exponential bound for the energy. We note however that the amount of energy outside is finite, in opposition with the problem of the section 3 where an exponential decay could not be reached.

By comparing equations (58) and (68), we note that the poloidal energy is decaying much slowly than the toroidal field. This can easily be explained: P is not vanishing outside and create a large amount of energy. In the vaccum, there is no mechanisms to dissipate the energy and this energy is forced to collapse on the conductor by use of the Poynting flux. In consequence it is more difficult for the system to dissipate this energy. In the limit of an infinite energy, we have seen in section 3 that the system lost its ability to exponentially damp the energy and we get a power law.

The conclusions presented here simply a conjecture of the different case explored here and can not be taken as rigourous assumption.We will next use a more simple model trying to capture the essential arguments presented here.

5 Simple model

The geometries and the calculations involved in the previous cases are relatively complicated. In this section we will study a very idealized mathematical model, in the perspective of a better comprehension of the different behavior of the energy depending on the situation outside the conductor.

Let us consider a one dimensional problem involving a scalar $\phi(x,t)$ depending only on the x direction. The domain is divided in two part: the inside part for $r < a$ and the outside part from a to infinity. Inside, ϕ obey the equation:

$$\dot{\phi} = D\phi'' \tag{69}$$

Where the dot means time derivative and the prime means x derivative. D is a coefficient of diffusion. Outside the conducting region, we suppose an equation of the type:

$$\phi(x)'' - m^2\phi = 0 \tag{70}$$

Where m is a parameter representing the spatial damping of the field outside. Indeed solution outside is of the form $\phi \sim e^{-m(x-a)}$. By supposing exponential decay for the time dependance of ϕ we get the equation :

$$\phi(x)'' + \frac{\alpha}{D}\phi = 0 \tag{71}$$

α is the decay rate of the mode. The boundary condition for the field at the origin is $\phi'(0) = 0$ A solution of this equation satisfaying ϕ non-zero at $x = 0$ is $\phi(x) = cos(kx)$ with $k = \sqrt{\alpha/D}$. We can now use the boundary conditions for this problem. We suppose here that ϕ and its derivative in x are continuous at the interface, leading to the same kind of relation than in the case of the poloidal field:

$$\phi' + m\phi = 0 \tag{72}$$

By plugging the expression for ϕ in this equation, we get:

$$tan(k_n a) = \frac{m}{k_n} \tag{73}$$

This yield a quantification of the decay rate $\alpha_n = k_n^2 D$. We can now expand our field on the Fourier-cosine series:

$$\phi(x,t) = \sum_n \phi_n cos(k_n x) e^{-k_n^2 Dt} \tag{74}$$

By using of the classical orthogonality relationship for cosine, the coefficients are obviously given by:

$$\phi_n = \int_O^\infty dx cos(k_n x)\phi(x,0) \tag{75}$$

As in the previous section, the energy is bounded by the less damped mode and we have:

$$\epsilon_\phi \leq \epsilon_\phi(0) e^{-2Dk_1^2 t} \tag{76}$$

This model show clearly how the decay rate of the energy of a field is related to the extension of this field in the space. Here, m represent the damping of the field outside the conductor. For large m, the field is strongly damped near the boundary and tend to a constant value of $(\frac{\pi}{2a})^2$ when m is increased. This is the situation for the diffusion of toroidal field seen before, where the field is confined to the inside region. When m is decrease, this correspond to an increase of the extension of the field outside the conductive region. The slowest decay rate is then reducing and the corresponding energy is less and less damped. In the limit of m very small, the decay rate of the energy tends to zero. However, one can note that when $m = 0$, the only solution satisfying the boundary conditions is $\phi = 0$. This behavior illustrate all the case studied before and show clearly the relationship between the external energy and the internal diffusion.

Using these simple examples, we have seen how the energy outside a conductor can play an essential role in the dissipation of the energy inside the conductor. In particular, a large amount of energy outside the conductor will in general collapse on the conductor and reduce the damping of the total energy by forcing the system to dissipate more energy. This means that surrounding a conductor with vaccum can help the system to keep its energy. In experimental situation, for instance in dynamo experiment, the conducting fluid is in general separated from the insulating region by the container, which can be a metal with different magnetic properties. In the next section we will study the decay problem for this more complicated situation.

6 Effect of ferromagnetic materials

We consider the same situation than for the decay problem in spherical geometry. In addition, we suppose now that there is two concentric sphere with region of different permeability. We consider only the decay problem for a poloidal magnetic field. The equation in the inner sphere of radius $r = r_1$ is given by:

$$\frac{\partial P}{\partial t} = \eta_1 \Delta P \tag{77}$$

For the external shell $(r_1 < r < r_2)$ we have the equation :

$$\frac{\partial P}{\partial t} = \eta_1 \Delta P \, r < a \tag{78}$$

For $r > r_2$, the field is harmonic in a medium of permeability μ_0 By using spherical harmonic expansion and solving the radial equation we find that the solution to this problem, for a given decay rate α, is given by:

$$P^l(r) = C r^{-1/2} J_{l+\frac{1}{2}} \left(\sqrt{\frac{\alpha}{\eta_1}} r \right) \tag{79}$$

for the field in the inner sphere. In the shell we have now:

$$P(r) = A r^{-1/2} J_{l+\frac{1}{2}} \left(\sqrt{\frac{\alpha}{\eta_2}} r \right) + B r^{-1/2} N_{l+\frac{1}{2}} \left(\sqrt{\frac{\alpha}{\eta_2}} r \right) \tag{80}$$

where $N_l(r)$ is the neumann function which appears because we can not invoke singularity at the center for eliminating it. Note that the decay rate is assumed to be identical in the two region of the problem. The boundary conditions at the external sphere is the same than before, but we suppose now that the permability of the two regions are not identical:

$$\frac{\partial P}{\partial r} + \frac{\mu_2}{\mu_0} \frac{(l+1)}{r_2} P = 0 \tag{81}$$

Using Bessel expansion it gives us the relation:

$$kr_2 J_{l-\frac{1}{2}}(kr_2)+(l+1)(\frac{\mu_2}{\mu_0}-1)J_{l+\frac{1}{2}}(kr_2) = -\frac{B}{A}\left[kr_2 N_{l-\frac{1}{2}}(kr_2)+(l+1)(\frac{\mu_2}{\mu_0}-1)N_{l+\frac{1}{2}}(kr_2)\right] \tag{82}$$

with $k=\sqrt{\frac{\alpha}{\eta_2}}$ The second boundary condition, acting on the inner core, consists on the following system:

$$[B]=0 \tag{83}$$

$$\left[\frac{\partial B}{\partial r}\right]=\frac{\mu_1}{\mu_2} \tag{84}$$

leading to the equation :

$$\frac{B}{A}=\frac{\sqrt{\frac{\mu_2\sigma_1}{\mu_1\sigma_2}}j_l(\rho_2)j_l'(\rho_1)-j_l'(\rho_2)j_l(\rho_1)}{-\sqrt{\frac{\mu_2\sigma_1}{\mu_1\sigma_2}}y_l(\rho_2)j_l'(\rho_1)-j_l'(\rho_2)j_l(\rho_1)} \tag{85}$$

The decay rates are then quantized and given by solving the system of equations (82) and (85).THis can be done numerically for any value of the permabilties. However, by considering the limit of high permeability for the shell, asymptotic calculation leads to the following expression for the decay rates of the system:

$$\alpha_l^n=(l+n)^2\frac{\pi^2\sigma_1\mu_1}{(r_1+r_2)^2\sigma_2\mu_2} \tag{86}$$

We see here that the limit of high permeability leads to a decrease of the damping rate of the energy of the system. This effect is stronger for a thin shell. This result suggest interesting modification for dynamo experiment in spherical geometry currently ongoing. For example, in the Madison experiment, surrounding the experiment with a shell of ferromagnetic materials (soft iron for instance) will probably yield a more excitable system and may cause a decreasing in the threshold of the dynamo instability(see figure **??**). Increasing the permeability of the inner sphere instead of the external ones leads to similar decrease of the decay rate and using ferromagnetic core would be a modification for dynamos experiment like DTS experiment in Grenoble in France or in Maryland experiment where two concentric sphere are used (see figure **??**).

In more realistic MHD situation, the conductor is a fluid, the system have now more complex way to dissipate the energy. It is thus of primary interest to consider the MHD case and see how the system dissipate the energy.

7 MHD bounds and background method

In this section we would like to study shear flow of a conducting fluid across a magnetic field. This situation occurs in many physical situation in astrophysical object or industrial application. In general, this imply the existence of Hartmann layers near the boundaries of the system. In particular we want to study such an Hartmann layer when one consider

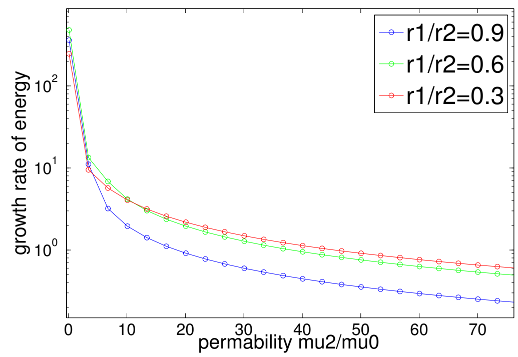

Figure 1: decay rate of the energy inside the sphere when the permeability of the shell is increased, for different aspet ratio. In the limit of ferromagnetic shell, the energy of the system do not decay.

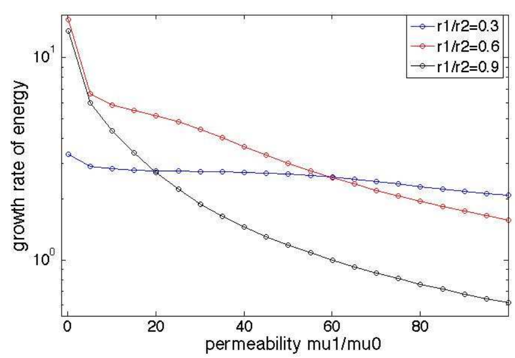

Figure 2: decay rate of the energy inside the sphere when the permeability of the core is increased, for different aspect ratio. In the limit of ferromagnetic core, the energy of the system do not decay.

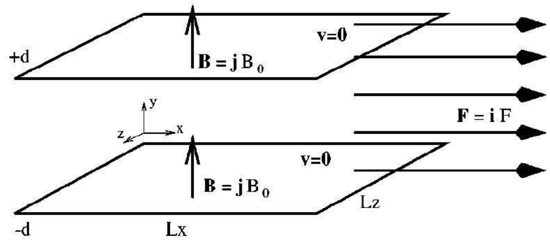

Figure 3: Setup of the Hartmann problem

realistic boundary conditions for the magnetic field, for instance when the conducting region is in contact with insulator walls or vaccum.The main quantities of interest are the energy stability and the rate of dissipation of this system.

The problem consist of a layer of conducting fluid of kinematic viscosity ν and magnetic resistivity η confined between two horizontal plates of size L in x and z directions. The plates are located between $y = -d/2$ and $y = +d/2$. There is a force F creating some flow in the x direction and an imposed constant magnetic field B_0 in the vertical direction. We assume periodic boundary conditions in the horizontal directions x and z for all the fields. In the vertical direction y there is no slip boundary conditions for the velocity field and the magnetic field match a potential field due to insulating region outside (see figure **??**).

The solenoidal velocity fields $\mathbf{v}$ and $\mathbf{B}$ satisify the MHD equations:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla\pi + \nu\Delta\mathbf{v} + \mathbf{F} + \frac{1}{\mu\rho}(\mathbf{B} \cdot \nabla)\mathbf{B}, \quad (87)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta\Delta\mathbf{B}. \quad (88)$$

In the above equations, ρ is the density, μ is the magnetic permeability and σ is the conductivity of the fluid.

A laminar solution for Hartmann flow is given by:

$$U_H = \frac{Fd}{2B_0}\sqrt{\frac{\eta}{\nu}}\left[\frac{cosh(\frac{B_0 d}{2\sqrt{\nu\eta}}) - cosh(\frac{B_0 y}{\sqrt{\nu\eta}})}{sinh(\frac{B_0 d}{2\sqrt{\nu\eta}})}\right] \quad (89)$$

for the velocity field and :

$$B_H = \frac{Fd}{2B_0}\left[\frac{sinh(\frac{B_0 y}{\sqrt{\nu\eta}})}{sinh(\frac{B_0 d}{2\sqrt{\nu\eta}})} - \frac{2y}{d}\right] \quad (90)$$

For the magnetic field.

We will use different dimensionless number in this section. The Hartmann number:

$$Ha = \frac{B_0 d}{\sqrt{\nu\eta}} \quad (91)$$

The grashoff number:

$$G_r = \frac{Fd^3}{\nu^2} \quad (92)$$

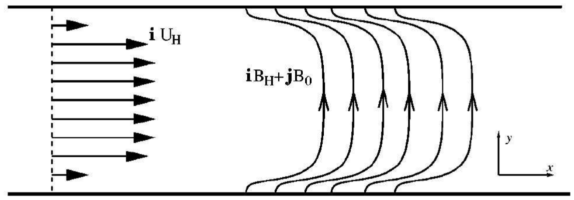

Figure 4: Laminar solution for the Hartmann problem

and the magnetic grashoff number:

$$G_m = \frac{Fd^3}{\nu\eta} \tag{93}$$

We define the dissipation associated with this laminar state as:

$$D = \nu\langle|\nabla \mathbf{U_H}|^2\rangle + \eta\langle|\nabla \mathbf{B_H}|^2\rangle \tag{94}$$

where brackets indicate space averaging. We can then evaluate the laminar dissipation:

$$D = \frac{Ha^{1/2}}{2\sqrt{2}Ha}\left[\coth(Ha) - \frac{1}{Ha}\right] \tag{95}$$

We note here that the magnetic field tends to zero when the applied field vanish and the velocity reduce to Poiseuille flow.

7.1 Background decomposition

In order to do the energy stability and estimate bounds on the dissipation, an useful method is to decompose the fields into two parts. A stationnary background profile depending only on y and a perturbation part:

$$\mathbf{v} = \mathbf{i}U_b(y) + \mathbf{u} \tag{96}$$

$$\mathbf{B} = \mathbf{i}B_b(y) + \mathbf{j}B_o + \mathbf{b} \tag{97}$$

The background profile and the perturbations can be any fields given the conditions that the total field have to be divergence-free and satisfy the boundary conditions. We can then use this decomposition in the equations (87) and (88). By adding the two equations, this gives the expression for the energy of the perturbation defined by $E = \mathbf{u}^2/2 + \mathbf{b}^2/2$:

$$\begin{aligned}\frac{\partial E}{\partial t} = \langle u_x B_0 B_b'\rangle + \langle b_x B_0 U_b'\rangle + \langle (b_x b_y - u_x u_y) U_b'\rangle \\ + \langle (u_x b_y - u_y b_x) B_b'\rangle + \eta\langle \Delta\mathbf{b}.\mathbf{b}\rangle - \nu\langle|\nabla \mathbf{u}|^2\rangle \\ + \nu\langle u_x U_b''\rangle + \eta\langle b_x B_b''\rangle + \langle u_x F\rangle\end{aligned} \tag{98}$$

7.2 Poincare inequality

Before doing the energy stability, we need to obtain an inequality for the term $\langle \Delta\mathbf{b}.\mathbf{b}\rangle$ ine the energy equation above. Indeed, the classical Poincare inequality which we use for the

velocity field can not apply for the magnetic field due to the boundary conditions.
Let's derive the magnetic field in Fourier space:

$$b_i = \sum_k b_i^k(y)\phi_k \tag{99}$$

i.e.

$$b_i = \sum_{k=0}^{\infty} b_i^k(y)e^{i(k_x x+k_z z)} \tag{100}$$

where subscipt i stands for x,y or z and ϕ_k are the eigenfunction of the horizontal laplacian with eigenvalue $\lambda_k = k_x^2 + k_z^2 = k_h^2$, satisfying the equation:

$$(\partial_{xx} + \partial_{zz})\phi_k = \lambda_k\phi_k \tag{101}$$

We see that the fourier coefficient depend on y since b is different from zero at the boundaries and Fourier decomposition is impossible in this direction. However, the field is harmonic outside the conductor and the Fourier modes satisfy :

$$(\partial_{yy} - k_h^2)b_i^k(y) = 0 \tag{102}$$

which give us :

$$b_i^k(y) = e^{-k_h(y-d/2)} \tag{103}$$

$$b_i^k(y) = e^{k_h(y+d/2)} \tag{104}$$

In the same manner that for the poloidal field in the sphere, the continuity of the field and its derivative imply :

$$\partial_y b_i^k(\frac{d}{2}) + k_h b_i^k(\frac{d}{2}) = 0 \tag{105}$$

$$\partial_y b_i^k(-\frac{d}{2}) - k_h b_i^k(-\frac{d}{2}) = 0 \tag{106}$$

We remark here that the magnetic field satisfy mode by mode exactly the same conditions than the toy model. Suppose now that the vetical dependance of the field can be described by the following expansion for the coefficents $b_i^k(y)$:

$$b_i^k(y) = Acos(q_k y) + Bsin(q_k y) \tag{107}$$

The boundary conditions becomes:

$$tan\left(\frac{q_k d}{2}\right) = \pm\left(\frac{k_h}{q_k}\right)^{\pm 1} \tag{108}$$

This equation yield an expression for the wavenumber q_k Using this expansion, it is straightforward to derive an expression for $\Delta B.B$:

$$-\langle \Delta B.B\rangle = -\int_{-\frac{d}{2}}^{+\frac{d}{2}} dy \sum_k b^k(\partial_{yy} - k_h^2)b^k$$
$$= \int_{-\frac{d}{2}}^{+\frac{d}{2}} dy \sum_k (q_k^2 + k_h^2)|b^k|^2) \geq (q_1^2 + k_1^2)\langle B^2\rangle \tag{109}$$

Where we use the first wavenumber to bound the expression. Eliminate the $k = 0$ terms, (there is no horizontally averaged magnetic field) we get the Poincare inequality :

$$-\eta\langle \Delta\mathbf{B}.\mathbf{B}\rangle \geq \eta(\frac{\pi^2}{L^2} + q_1^2)\langle B^2\rangle \tag{110}$$

7.3 Energy stability

The stability of the Hartmann layer is an important characteristic of the system. A complete determination of the problem would imply to solve the full variationnal problem involving the resolution of the Euler-Lagrange equation. We will instead derive rigourous energy stability by considering appropriate bounds on the energy. Note that this method generally capture the main behavior of the system and the full resolution only improve the results by some factor.

The background kinetic and magnetic fields are now taken to be the laminar solutions of the problem. The energy equation (98) simplify in :

$$\frac{\partial E}{\partial t} = -F[\mathbf{u}, \mathbf{b}] \tag{111}$$

where the funtional F is given by:

$$\begin{aligned} F[\mathbf{u}, \mathbf{b}] = \langle (b_x b_y - u_x u_y)U'_H\rangle + \langle (u_x b_y - u_y b_x)B'_H\rangle \\ - \eta\langle \Delta\mathbf{b}.\mathbf{b}\rangle + \nu\langle |\nabla\mathbf{u}|^2\rangle \end{aligned} \tag{112}$$

Using Holster and Young inequality, we can bound the two first term of the above quaratic form as follow:

$$\langle (b_x b_y - u_x u_y)U'_H\rangle \leq \frac{max(|U'_H|)}{2}\langle \mathbf{b}^2 + \mathbf{u}^2\rangle \tag{113}$$

$$\langle (u_x b_y - u_y b_x)B'_H\rangle \leq \frac{max(|B'_H|)}{2}\langle \mathbf{b}^2 + \mathbf{u}^2\rangle \tag{114}$$

The two last term can be controlled by Poincare inequalities

$$\nu\langle |\nabla\mathbf{u}|^2\rangle \geq \nu\frac{\pi^2}{d^2}\langle \mathbf{u}^2\rangle \tag{115}$$

$$-\eta\langle \Delta\mathbf{b}.\mathbf{b}\rangle \geq \eta(\frac{\pi^2}{L^2} + q_0^2)\langle \mathbf{b}^2\rangle \tag{116}$$

The above inequality yield conditions for non-negativity of the quadratic form (112) which can be formulated as:

$$G_m^2\left[Ha\coth(Ha) - 1\right]^2 \leq Ha^2\left(2\pi^2 - G_r\right)\left(2\left(\frac{d^2\pi^2}{L^2} + q_0^2 d^2\right) - G_m\right) \tag{117}$$

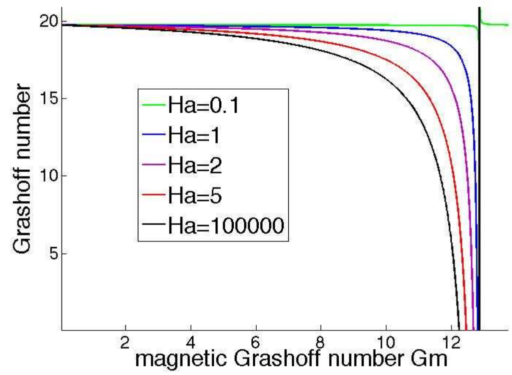

Figure 5: Energy stability of the Hartmnn flow for different Hartmann number.

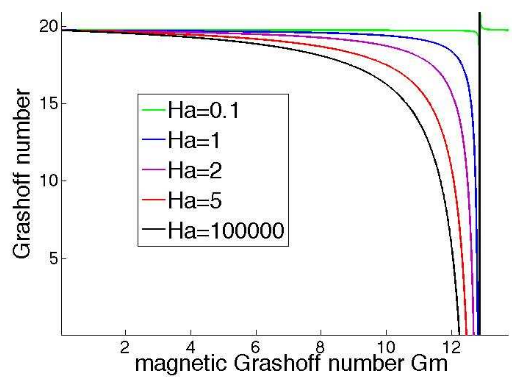

Figure 6: Evolution of the stability with the aspect ratio of the problem.

In the limit where G_m tends to zero, we find the energy stability of the Poiseuille flow. Figure (**??**) show the energy stability in the plane $G_r - G_m$ for different Hartmann number. We see here that the results depend now on the aspect ratio $r = L/d$ and the figure (**??**) illustrate the destabilisation of the Hartmann flow when the lenght perpendicular to the magnetic field is increased.

7.4 Bound on dissipation rate

Using the procedure followed in the previous section, we are now interesting in the total dissipation of the system, defined by:

$$D = \nu \langle |\nabla \mathbf{v}|^2 \rangle + \eta \langle |\nabla \mathbf{B}|^2 \rangle \tag{118}$$

To determined bounds on the dissipation, we now choose arbitrary backgrounds, which are not the laminar solutions but which will be determined in order to get the best possible estimation of the bound.
Taking the equation (98) and adding to the energy half of the total dissipation we obtain:

$$\frac{\partial E}{\partial t} + \frac{D}{2} = \langle u_x B_0 B_b' \rangle + \langle b_x B_0 U_b' \rangle + \langle (b_x b_y - u_x u_y) U_b' \rangle$$
$$+ \langle (u_x b_y - u_y b_x) B_b' \rangle - \frac{\eta}{2} \langle \mathbf{\Delta b} . \mathbf{b} \rangle - \frac{\nu}{2} \langle |\nabla \mathbf{u}|^2 \rangle$$
$$+ \frac{\nu}{2} \langle U_b'^2 \rangle + \frac{\eta}{2} \langle B_b'^2 \rangle + \langle u_x F \rangle \quad (119)$$

In order to eliminate the linear term in the previous expression, we perform the following change of variable:

$$\mathbf{u} = \mathbf{w} - \mathbf{i} V_b(y) \quad (120)$$

$$\mathbf{b} = \mathbf{h} - \mathbf{i} H_b(y) \quad (121)$$

with:

$$\nu V_b'' = B_0 B_h' \quad (122)$$

and:

$$\eta H_b'' = B_0 U_h' \quad (123)$$

We end up with the final expression for the dissipation rate of the Hartmann layer:

$$D = 2F \langle U_b \rangle - D_b + Q_b \quad (124)$$

where the functional Q is given by

$$Q_b = \nu \langle |\nabla \mathbf{w}|^2 \rangle - \eta \langle \mathbf{\Delta h} . \mathbf{h} \rangle - 2 \langle (h_x h_y - w_x w_y) U_b' + (w_x h_y - w_y h_x) B_b' \rangle \quad (125)$$

We see that if we can find background fields such that the quadratic form Q_b is non-negative, the dissipation will be bounded by an expression depending only on the backgroun profiles.

Using the same inequalities than for the estimation of the energy stability, we can obtain conditions for positivity of the quadratic form Q_b leading to a lower bound for the dissipation . This bound is valid only in the region of parameter space defined by

$$Rm \leq 4 \left(\frac{\pi^2 d^2}{L^2} + d^2 q_0^2 \right) \frac{1}{Re} \quad (126)$$

One can note that this prediction on the dissipation rate of the system is very weak. Indeed, the bound on the dissipation is restricted to a laminar region and do not apply for turbulent behavior. This problem come from the fact that the estimation of the bounds is done without taking into account the presence of the boundary layers which would occur in any physical situation.
Focusing on the boundary of the system, we can use the fundamental theorem of calculus, Schwartz and Young inequalities in order to get:

$$\langle (w_x w_y) U_b' \rangle \leq U^* \delta |\nabla \mathbf{w}|^2 \rangle \quad (127)$$

The derivation of this equation come from the fact that the conditions of vanishing perturbation w on the boundary give a control on the magnitude of the gradient of the field inside the conductor. The same derivation is not possible for the magnetic field, where the matching with an external potential field do not offer any control on the value of h at the boundary. In fact, these boundary conditions specify only a relation between the field and its gradient at the boundary, which is not sufficient to conclude something about the gradient inside the conductor. In consequence, it is not possible to characterize the behavior of the magnetic field in the boundary layers, leading to the impossibility of obtaining a bound on the dissipation rate.

8 Conclusion

We have seen in the first sections that the vaccum can play an important role in the dissipation of a conductor or MHD system. It is thus of primary interest to see how stability or dissipation rate are changed when one take into account realistic boundary conditions. Using the background we studied the Hartmann flow problem in order to derive energy stability and lower bound on the dissipation rate. We have seen that the insulating boundary conditions do not allow sufficient control on the magnetic field in the boundary layer to get a bound on the dissipation rate. This suggest to use a different approach, involving a modification of the classical background method.