Lecture 7: Oceanographic Applications.

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1 Introduction

Nonlinear waves can be studied by a number of models, which include the Korteweg–de Vries (KdV) and the Kadomtsev-Petviashvili (KP) equations. These equations are reviewed briefly and applied here to investigate the dynamics of surface waves in the ocean. The resulting tsunami of the Indian Ocean earthquake in 2004 are considered as a specific example. The tsunami dynamics, as well as wave patterns observed near shore, are explained by ideas developed in the previous lectures.

2 Review of waves in shallow water

A theory of nonlinear surface waves in shallow water was presented in Lectures 5 and 6. Relevant aspects of this theory are reviewed first for later discussion of observed waves in the ocean. Note that the theory, which includes effects due to dispersion, is different from the hyperbolic partial differential equations called the shallow-water equations, which describe non-dispersive waves and are presented in Lecture 8.

Consider the water in the ocean as an incompressible, irrotational fluid with velocity potential ϕ. A Cartesian coordinate system is adopted with the x and y axes in the horizontal plane and the z axis pointing upwards from the mean level of the fluid. The fluid lies in the domain bounded below by a prescribed topography, $z = -h(x, y)$, and above by a free surface to be determined, $z = \eta(x, y, t)$. In this theoretical framework, the governing equations at any time t are given by

$$
\begin{aligned}
\nabla^2\phi &= 0 \qquad -h < z < \eta, && (1)\\
\frac{\partial \eta}{\partial t} + \nabla\phi\cdot\nabla\eta &= \frac{\partial \phi}{\partial z} \qquad z = \eta, && (2)\\
\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + g\eta &= 0 \qquad z = \eta, && (3)\\
\mathbf{n}\cdot\nabla\phi &= 0 \qquad z = -h. && (4)
\end{aligned}
$$

The velocity potential satisfies the incompressibility condition (1), subject to the boundary conditions at the top and bottom of the domain. Equations (2) and (3) are respectively the kinematic and dynamic conditions on the free surface. Gravity g is dominant and surface tension is neglected in the dynamic condition (3), implicitly restricting the analysis to waves with horizontal lengthscales much larger than the capillary length scale. Equation

(4), where $\mathbf{n}$ is the unit vector normal to the topography, is the impermeability condition at the rigid bottom of the ocean.

A theory of nonlinear waves in shallow water is developed by introducing the following approximations. First, the characteristic variation a in η is small compared to the entire depth of the water h, under the small-amplitude approximation. Second, waves propagate in the x direction with a typical wavelength L_x, much longer than the depth of the water. This is the shallow water or the long wave approximation. Third, the motion is nearly one-dimensional, provided the horizontal length scale L_y in the transverse direction of the propagating wave is much longer than L_x. It is assumed that all small effects balance by order of magnitude such that $a/h = O(\epsilon)$, $(h/L_x)^2 = O(\epsilon)$ and $(L_x/L_y)^2 = O(\epsilon)$ or smaller, where $\epsilon \ll 1$ is the small parameter in the problem. See Lectures 3, 5 and 6 for the derivation and discussion of these scalings.

At leading order in ϵ, η satisfies the one-dimensional wave equation given by

$$\frac{\partial^2 \eta}{\partial t^2} = c^2 \frac{\partial^2 \eta}{\partial x^2}, \tag{5}$$

for constant topography h, where $c = \sqrt{gh}$ is the phase speed of the wave. The solution is a linear combination of traveling waves given by

$$\eta = \epsilon h[f(x - ct; \epsilon y, \epsilon t) + F(x + ct; \epsilon y, \epsilon t)] + O(\epsilon^2), \tag{6}$$

where f and F are amplitudes of the waves that propagate in the positive and negative x directions respectively. At next order in ϵ, either the KdV or the KP equation is obtained, depending on whether $(L_x/L_y)^2 \ll O(\epsilon)$ or $(L_x/L_y)^2 = O(\epsilon)$ respectively. Waves propagating along the x axis with $(L_x/L_y)^2 \ll O(\epsilon)$, are one-dimensional and described by the KdV equation [8]

$$\frac{\partial f}{\partial \tau} + f\frac{\partial f}{\partial \xi} + \frac{\partial^3 f}{\partial \xi^3} = 0, \tag{7}$$

where $\tau = \epsilon t$ is the slow time variable and $\xi = x - ct$ is the spatial coordinate in the reference frame of the moving waves. The KdV equation indicates that the wave amplitude evolves due to nonlinear and dispersive effects, corresponding to the second and third terms of (7) respectively. Two-dimensional waves, with $(L_x/L_y)^2 = O(\epsilon)$, disperse weakly in the transverse direction of propagation and are described by $f(\xi, \zeta, \tau)$ satisfying the KP equation [6] instead

$$\frac{\partial}{\partial \xi}\left(\frac{\partial f}{\partial \tau} + f\frac{\partial f}{\partial \xi} + \frac{\partial^3 f}{\partial \xi^3}\right) + \frac{\partial^2 f}{\partial \zeta^2} = 0, \tag{8}$$

where $\zeta = \epsilon y$ is the slowly-varying coordinate in the y direction. Both KdV and KP equations are integrable, meaning that they admit soliton solutions. A soliton is a special type of solitary wave, a localized wave that travels without any change in shape or size. The soliton has a permanent form in structure even after interacting with another oncoming soliton. Solitary waves occur frequently because they represent the long-time limit of waves that arise from a range of initial conditions. The possible appearance of solitary waves in the ocean is examined below.

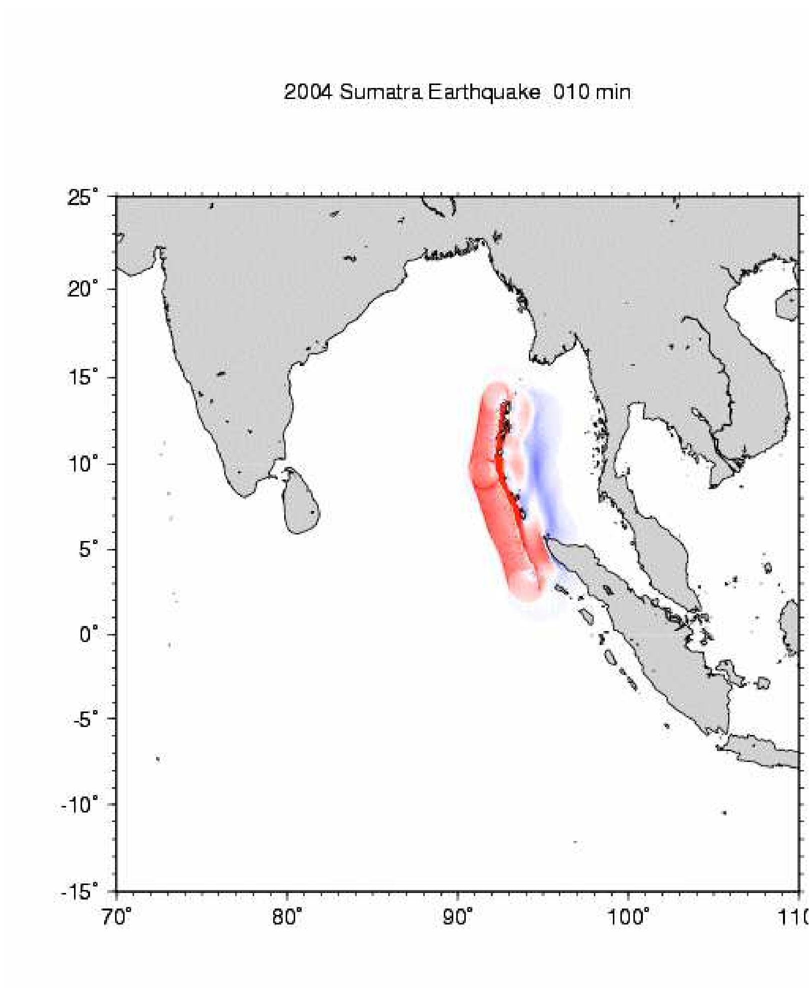

Figure 1: Numerical simulation of surface elevation (red) and depression (blue) of the Indian Ocean, soon after a series of undersea earthquakes occurred off the coast of Sumatra over an interval of 10 minutes on 26 December, 2004. An animation of the evolving surface of the ocean is available at http://staff.aist.go.jp/kenji.satake/.

Variable (units)	Magnitude
h (km)	3.5
a (km)	0.001
L_x (km)	100
L_y (km)	1000
c (km/h)	650
u (km/h)	0.2

Table 1: Typical magnitudes of the tsunami triggered on 26 December, 2004, in the Indian Ocean of depth h. Surface waves with amplitude a, wavelength L_x and width L_y traveled with phase speed c and fluid speed u.

3 Application to Tsunami waves

The sudden displacement of a large volume of water results in a series of surface waves in the ocean, called a tsunami. A famous example is the tsunami caused by a series of undersea earthquakes off the western coast of Sumatra on 26 December, 2004. These earthquakes occurred near-simultaneously along a 1000 km fault line. The surface elevation and depression of the ocean, soon after the earthquake, have been reproduced numerically as shown in Figure 1. The resultant tsunami waves devastatingly struck the coasts of the Indian Ocean and caused many casualties.

The magnitude of the tsunami in the Indian Ocean is estimated as presented in Table 1. Note that a/h, $(h/L_x)^2$ and $(L_x/L_y)^2$ are all small with a common order of magnitude ϵ in the narrow range from 10^{-2} to 10^{-3}, consistent with the approximations required to obtain solitary waves governed by the KP equation. However, we know from Lectures 5 and 6 that solitary waves only develop on the long time scale of order $1/\epsilon$, namely 10^2 to 10^3 hours in this case. Given that the tsunami traveled a distance of approximately 1500 km across the Bay of Bengal with phase speed 650 km/h, the initial displacement of water did not have sufficient time to evolve into a solitary wave. The tsunami would have needed to propagate a distance two or three orders of magnitude longer than across the Bay of Bengal to develop into a solitary wave, as governed by the KdV or the KP equation.

In contrast to the short distance traveled by the tsunami in the Indian Ocean, the tsunami triggered off the coast of Chile on 22 May, 1960, by the most powerful earthquake ever recorded, may have developed into solitary waves. The tsunami propagated tens of thousands of kilometers across the Pacific Ocean and reached the coast of Japan after 22 hours. It is possible that this tsunami developed into solitary waves, which propagated without change in their structural form.

In the deep and open ocean, the dynamics of tsunami waves are characterized by small amplitudes and long wavelengths, which are hardly detected by an observer on a boat on the surface. As the tsunami approaches near shore, the depth of the water decreases, resulting in a decelerating wave speed at the front while the back of the tsunami maintains speed. The waves consequently compress horizontally and grow vertically in a process called wave shoaling. The waves may grow to tens of meters in amplitude, causing much damage when they reach and strike coastlines.

The tsunami caused by the Sumatra earthquake in 2004 propagated eastwards to the

Figure 2: The tsunami of 26 December, 2004, approaching Hat Ray Leah beach on the Krabi coast, Thailand [1].

coast of Thailand with a wave of depression at the front. This reflects the downward displacement of water on the eastern side of the area where the tsunami originated, as shown in Figure 1. As shown in Lecture 5, laboratory experiments demonstrate that a downward displacement of water leads to a wave-train preceeded by a wave of depression[5]. When the front of the wave of depression arrived in Thailand, the water along the shoreline receded dramatically and exposed areas that are otherwise submerged, as shown in Figure 2. Soon after, successive waves of large amplitude struck the coast and destructed the area.

Risks posed to coastlines can be assessed by considering the tsunami in the deep ocean, from initiation to propagation. Tsunami is generated by a thrust fault, a normal fault or a landslide. A crucial quantity for estimating the size of the tsunami is the volume of water displaced by seismic events under the water. The time for the tsunami to propagate between two given positions, x_1 and x_2, is estimated by minimizing

$$\int_{x_1}^{x_2} \frac{ds}{c(s)} \tag{9}$$

over all possible paths from x_1 to x_2. Note that the shortest distance from x_1 to x_2 may not be the path that minimizes (9) because the wave speed $c(s) = \sqrt{gh(s)}$ may increase considerably with position s along another path.

It remains a challenge to predict the detailed dynamics of tsunami, particularly near the shore. As the tsunami approaches the coast, the dynamics are influenced by effects due to reflection, refraction and breaking of the waves. The near-shotre shape, size and speed of the tsunami are still poorly understood.

4 Oscillatory waves in shallow water

In this Section, further insight into waves in shallow water is provided by examining another special class of waves, which oscillate periodically. Indeed, in contrast to tsunami waves

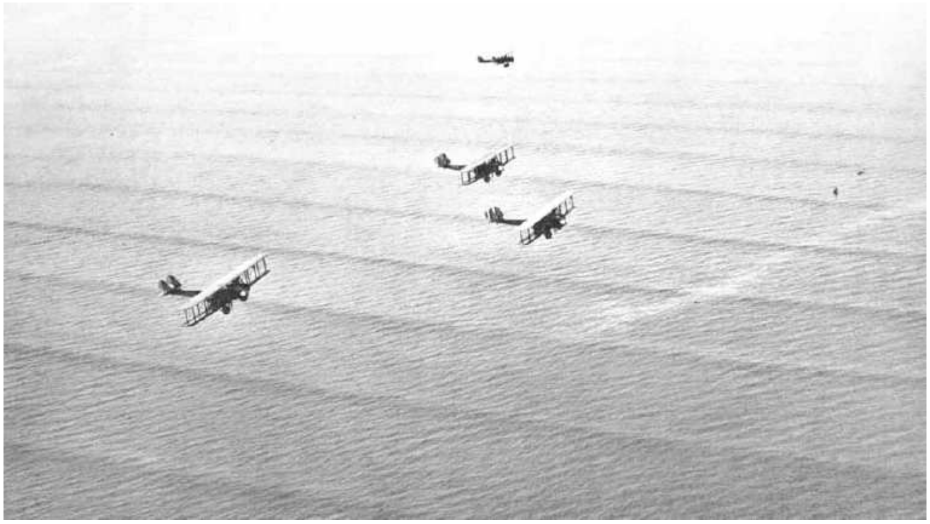

Figure 3: Cnoidal waves traveling in the ocean below flying aircrafts (National Geographic, 1933).

which are caused by earthquakes and landslides, most surface waves in the ocean are caused by storms and winds. These waves oscillate periodically and may develop patterns of permanent form, which are presented below.

The simplest model of long waves of small amplitude features linear, non-dispersive waves governed by (5), which all travel with speed $\sqrt{gh}$. For long waves of moderate amplitude, all traveling in approximately the same direction in water of uniform depth, a better approximation is the KP equation (8). It can be shown that the KP equation is completely integrable and admits solutions of the form[7]

$$f(\xi, \zeta, \tau) = 12 \frac{\partial^2}{\partial \xi^2} \log \Theta, \tag{10}$$

where Θ is a Riemann theta function of genus G. The genus is an integer corresponding to the number of independent phases in the solution. For example, solutions of genus 1 are the one-dimensional and periodic cnoidal wave solutions of the KdV equations discussed in Lectures 3, 5, and 6. Real cnoidal waves are shown in Figure 3; they propagate in the direction perpendicular to the wave crests with a coherent and permanent structure.

As discussed in Lecture 4, complete integrability guarantees the existence of quasi-periodic solutions obtained from (10), such as two-phase solutions of the KP equation. They are described by (10) with a Riemann theta function Θ of genus 2. The surface patterns are quasi-periodic in the sense that they cannot be characterized by a single period. The patterns are hexagonal in shallow water, as reproduced in Figure 4. The two-phase solutions can be interpreted as a combination of two cnoidal waves that meet at an angle, where a shift in phase occurs due to their interaction. The two-phase solutions agree remarkably well with surface patterns observed in laboratory experiments, demonstrating that the KP equation accurately describes real phenomena [3]. The excellent agreement between the theory and the experiments supports the existence of permanent patterns in the ocean, such as the approximately periodic patterns observes along the shoreline of Duck, North Carolina, as shown in Figure 5.

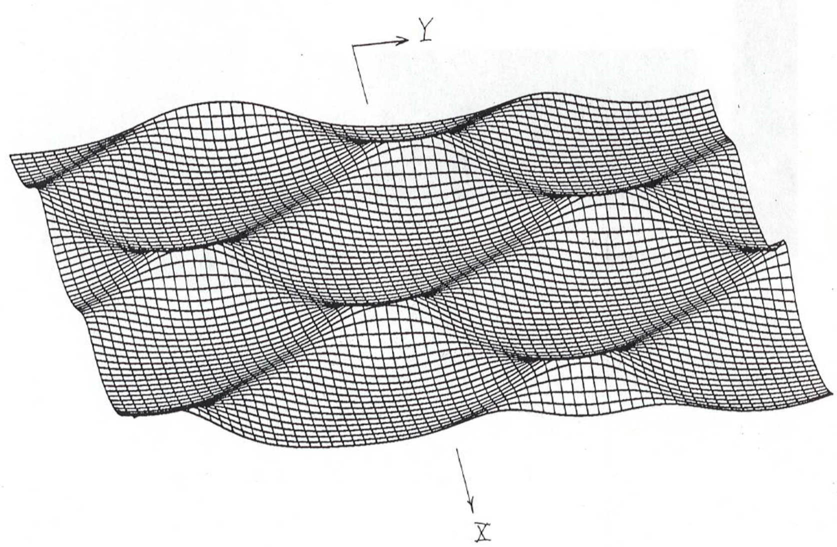

Figure 4: Two-phase solution of the KP equation propagating in the x direction with weak dispersion in the y direction [4].

Figure 5: Nearly periodic ocean waves with two-dimensional patterns along the shoreline of Duck, North Carolina.

Several open problems remain regarding these periodic wave patterns. Although it has been proved that they should exist in water of *any* depth [2], only the simplest patterns have ever been discovered. The KP equation admits other periodic solutions of permanent form with genus $G > 2$, which have not been explored yet. Another fundamental unsolved problem concerns the stability of these two-dimensional wave patterns of permanent form. A stability analysis of the patterns could determine whether one should expect to encounter them in nature or not, and how frequently. Another important problem is the effect of variable topography on the wave patterns, as is the case near shore. A better understanding of these problems could provide useful insight into the dynamics of surface waves in the ocean.

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