

Lecture 5: Waves in shallow water, part I: the theory

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1 Introduction

We saw in Lecture 2 a linearized model of water waves, in which all waves (necessarily) have very small amplitude, and the longest waves (with wavenumbers near $k = 0$) propagate with essentially no dispersion. The next step towards a more realistic description of water waves is to introduce weakly dispersive and weakly nonlinear effects. The derivation of the Korteweg – de Vries (KdV) equation, which follows this approach, is the topic of the first part of this lecture. The properties of the KdV equation are presented in a second part, followed by a third part which discusses the accuracy of this equation for water waves in real-life settings and its relevance as an approximation of the water wave equations.

Let us recall the notations used so far:

- Cartesian coordinate system of variables (x, y, z) , the origin of z being the mean altitude of the free-surface,
- position of a particle at a given time: $\mathbf{x} = (x(t), y(t), z(t))$
- position of the free-surface: $z = \eta(x, y, t)$,
- position of the bottom: $z = -h(x, y)$,
- fluid velocity in the interior of the fluid: $\mathbf{u}(x, y, z, t) = (u, v, w)$,
- density of water: ρ ,
- surface tension: σ .

Let us also recall the approximations made so far, which still apply in this study:

- water is incompressible ($\nabla \cdot \mathbf{u} = 0$),
- uniform density ($\rho(x, y, z, t) = \rho$),
- no rotation,
- constant gravity vector pointing downwards,
- no viscous effects,
- irrotational flow ($\nabla \times \mathbf{u} = 0$), allowing us to define a velocity potential ϕ as $\mathbf{u} = \nabla\phi$.

The four equations of motion for the waves under these assumptions are:

$$\partial_t \eta + \nabla \phi \cdot \nabla \eta = \partial_z \phi \quad \text{on } z = \eta(x, y, t), \quad (1)$$

$$\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + g \eta = \frac{\sigma}{\rho} \nabla \cdot \left\{ \frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}} \right\} \quad \text{on } z = \eta(x, y, t), \quad (2)$$

$$\nabla^2 \phi = 0 \quad \text{for } -h(x, y) < z < \eta(x, y, t), \quad (3)$$

$$\partial_z \phi + \nabla \phi \cdot \nabla h = 0 \quad \text{on } z = -h(x, y). \quad (4)$$

2 Derivation of the KdV equation

The following derivation of the KdV equation uses a general method to derive approximate models of water waves, such as the Kadomtsev – Petviashvili (KP) equation or the Nonlinear Schrödinger (NLS) equation. It consists of four steps starting from what is considered as the exact set of equations of the problem. These four steps reflect the outline of this section:

1. to identify an appropriate limit of the problem. As mentioned earlier, we consider here 1D, long, weakly nonlinear waves.
2. to scale the equations in order to make these limits explicit,
3. to expand these equations in order to solve them approximately, order by order,
4. to introduce multiple scales as needed and solve the equations correctly.

Useful references for this section are Ablowitz & Segur [1] and Dauxois & Peyrard [7].

2.1 Limit of interest for KdV (or KP)

In this problem, a is the amplitude scale of η and L_x the characteristic scale along x . Five assumptions are made:

1. the propagation is along x only, implying that the partial derivatives along y are equal to zero. If it is only approximately true, then one can obtain the KP equation instead of KdV (see Lecture 3).
2. flat topography ($h = \text{constant}$),
3. shallow water (or long wave or weak dispersion) approximation: the horizontal scale of the waves is large compared with the mean water height. We then define δ such that $h = \delta L_x$ with $\delta \ll 1$,
4. weak nonlinearity: a is small compared to h but finite. We define ε such that $a = \varepsilon h$ with $\varepsilon \ll 1$,

5. weak nonlinear and weak dispersive effects balance: in the KdV case, it implies $\varepsilon = O(\delta^2)$.

The final point requires some clarification. The procedure outlined above in points 1 through 4 introduces two small parameters, ε and δ . Both are dimensionless and assumed to be small compared with unity, but little is known a priori on how they compare in size with each other. An “honest” procedure from here on would be to carry out an expansion of the original equations in both parameters ε and δ and deduce the correct scaling between the two numbers for which weakly nonlinear and weakly dispersive effects are comparable in size at the first non-linear order in the expansion. This method is unfortunately mathematically cumbersome.

For the sake of clarity, in these lectures, we use previous results which yielded the correct scalings, namely $\varepsilon = O(\delta^2)$. This assumption is revisited and discussed in Section 2.4. Hence, in the rest of the lecture, we set $\varepsilon = \delta^2$ and only retain ε as the scaling parameter.

2.2 Variable scaling

As announced at the beginning of this section, we introduce new non-dimensional variables (to be recognized by *) in order to make the previous assumptions explicit. The vertical coordinate is scaled by the total water height:

$$z = h z^*.$$

Following Section 2.1, we now rescale the horizontal variable with $L_x = h/\delta$ and the amplitude of the perturbation η with $a = \varepsilon h$, and use the identity $\varepsilon = \delta^2$:

$$x = L_x x^* = \frac{h x^*}{\sqrt{\varepsilon}}, \quad \eta = a \eta^* = \varepsilon h \eta^*.$$

The linear phase speed of infinitely long waves \sqrt{gh} can be used to define a characteristic speed. As the motion is driven by the free-surface and as the free-surface motion is very gentle, the characteristic fluid velocity is also very gentle compared to the phase speed of the linear wave. We choose the ratio of the two typical velocities to be of $O(\varepsilon)$, and carry this scaling over to the velocity potential:

$$u = u^* \varepsilon \sqrt{gh} \quad \Rightarrow \quad \phi = h \sqrt{\varepsilon gh} \phi^*.$$

Finally, a characteristic timescale is naturally defined as the characteristic period of the linear waves:

$$T = \frac{L_x}{\sqrt{gh}} = \sqrt{\frac{h}{g\varepsilon}} \quad \Rightarrow \quad t^* = \sqrt{\frac{g\varepsilon}{h}} t.$$

In the new coordinate system, the solutions of the problem (η^* , u^* and ϕ^* for example) are now functions of the re-scaled variables x^* and t^* , and of the parameter ε , e.g.

$$\eta^* = \eta^*(x^*, t^*; \varepsilon), \quad \phi^* = \phi^*(x^*, z^*, t^*; \varepsilon), \quad u^* = u^*(x^*, t^*; \varepsilon), \dots$$

2.3 Solve equations approximately, order by order

2.3.1 Approximate velocity field

Let us go back to the original function ϕ for a moment and write ϕ as a convergent Taylor series near the bottom boundary (i.e. a series in $z + h$) :

$$\phi(x, z, t) = \sum_{n=0}^{\infty} \frac{(h+z)^n}{n!} \phi^{(n)}(x, t) \quad (5)$$

where we use the notation $\phi^{(n)}(x, t) = \partial^n \phi / \partial z^n |_{x, t, z=-h}$. If we take the Laplacian of the last equation and use (3), we find that:

$$\nabla^2 \phi = \sum_{n=0}^{\infty} \frac{(h+z)^n}{n!} \left[\frac{\partial^2 \phi^{(n)}}{\partial x^2} + \phi^{(n+2)} \right] = 0. \quad (6)$$

Each term of the sum being independent, in order to satisfy this equation, one has to make sure that for each n , and at each point (x, t) ,

$$\frac{\partial^2 \phi^{(n)}}{\partial x^2} + \phi^{(n+2)} = 0. \quad (7)$$

Equation (4) tells us that $\phi^{(1)} \equiv 0$ since $\partial \phi / \partial z = 0$ at $z = -h$ for a flat bottom boundary. As a consequence, it follows from equation (7) that for each n being odd, $\phi^{(n)}(x, t) \equiv 0$. Let's define $\phi_0(x, t) \equiv \phi^{(0)}$. Equation (7) can be used recursively starting from $n = 0$ to find all of the functions $\phi^{(n)}(x, t)$ for n even:

$$\phi^{(2m)}(x, t) = (-1)^m \frac{\partial^{2m} \phi_0}{\partial x^{2m}}.$$

We can now rewrite ϕ as:

$$\phi(x, z, t) = \phi_0(x, t) + \sum_{m=1}^{\infty} \frac{(-1)^m}{(2m)!} (h+z)^{2m} \frac{\partial^{2m} \phi_0}{\partial x^{2m}}$$

$$\text{or as } \phi = h \sqrt{\varepsilon g h} \left[\phi_0^*(x^*, t^*; \varepsilon) + \sum_{m=1}^{\infty} \frac{(-\varepsilon)^m}{(2m)!} (1+z^*)^{2m} \frac{\partial^{2m} \phi_0^*}{\partial (x^*)^{2m}} \right] \quad (8)$$

in terms of the rescaled variables. We have not made any extra approximation so far, meaning that equation (8) provides an exact form of ϕ^* (provided the series converges).

If we differentiate equation (8) along the two directions and only retain the terms of the first two orders, we find:

$$u(x, z, t) = \varepsilon \sqrt{g h} \left[\hat{u}^*(x^*, t^*; \varepsilon) - \frac{\varepsilon}{2} (1+z^*)^2 \frac{\partial^2 \hat{u}^*}{\partial (x^*)^2} + O(\varepsilon^2) \right] \quad (9)$$

where we defined $\hat{u}^* = \partial \phi_0^* / \partial x^*$,

$$w(x, z, t) = \sqrt{\varepsilon g h} w^* = -\varepsilon \sqrt{\varepsilon g h} \left[(1+z^*) \frac{\partial \hat{u}^*}{\partial x^*} - \frac{\varepsilon}{6} (1+z^*)^3 \frac{\partial^3 \hat{u}^*}{\partial (x^*)^3} + O(\varepsilon^2) \right] \quad (10)$$

2.3.2 At the free surface

We now focus on equations (1) and (2). They both involve η and ϕ , or in other words $\eta^*(x^*, t^*; \varepsilon)$ and $\phi^*(x^*, z^*, t^*; \varepsilon)$. The other variable to be solved for is $\hat{u}^*(x^*, t^*; \varepsilon)$. We only have to solve these equations order-by-order and for that purpose, we expand η^* and \hat{u}^* in power series in ε :

$$\eta^*(x^*, t^*; \varepsilon) = \eta_1(x^*, t^*; \varepsilon) + \varepsilon \eta_2(x^*, t^*; \varepsilon) + O(\varepsilon^2) \quad (11)$$

$$\text{and } \hat{u}^*(x^*, t^*; \varepsilon) = u_1(x^*, t^*; \varepsilon) + \varepsilon u_2(x^*, t^*; \varepsilon) + O(\varepsilon^2) \quad (12)$$

Equation (1), in non-dimensional form, is:

$$\varepsilon \frac{\partial \eta^*}{\partial t^*} + \varepsilon^2 u^* \frac{\partial \eta^*}{\partial x^*} = w^* \quad \text{on } z^* = \varepsilon \eta^*. \quad (13)$$

Using equations (10), (11) and (12) to expand (13) in a power series of ε until $O(\varepsilon^2)$, we find:

$$\varepsilon \frac{\partial \eta_1}{\partial t^*} + \varepsilon^2 \frac{\partial \eta_2}{\partial t^*} + \varepsilon^2 u_1 \frac{\partial \eta_1}{\partial x^*} = -\varepsilon \frac{\partial u_1}{\partial x^*} - \varepsilon^2 \frac{\partial u_2}{\partial x^*} - \varepsilon^2 \eta_1 \frac{\partial u_1}{\partial x^*} + \frac{\varepsilon^2}{6} \frac{\partial^3 u_1}{\partial (x^*)^3}. \quad (14)$$

Note that we have kept the multiplicative ε factor everywhere to emphasize the order to which the equation is valid.

We also have to expand the dynamical equation of the free-surface (2). In non-dimensional variables, this equation reads:

$$\varepsilon \frac{\partial \phi^*}{\partial t^*} + \frac{\varepsilon}{2} (\varepsilon (u^*)^2 + (w^*)^2) + \varepsilon \eta^* = \varepsilon^2 \frac{\sigma}{\rho g h^2} \frac{\partial}{\partial x^*} \left\{ \frac{\frac{\partial \eta^*}{\partial x^*}}{\sqrt{1 + \varepsilon^3 \left| \frac{\partial \eta^*}{\partial x^*} \right|^2}} \right\} \quad \text{on } z^* = \varepsilon \eta^* \quad (15)$$

If we differentiate the latter along x^* to eliminate ϕ^* , we get

$$\varepsilon \frac{\partial u^*}{\partial t^*} + \frac{\varepsilon^2}{2} \frac{\partial (u^*)^2}{\partial x^*} + \frac{\varepsilon}{2} \frac{\partial (w^*)^2}{\partial x^*} + \varepsilon \frac{\partial \eta^*}{\partial x^*} = \varepsilon^2 \frac{\sigma}{\rho g h^2} \frac{\partial^2}{\partial (x^*)^2} \left\{ \frac{\frac{\partial \eta^*}{\partial x^*}}{\sqrt{1 + \varepsilon^3 \left| \frac{\partial \eta^*}{\partial x^*} \right|^2}} \right\} \quad \text{on } z^* = \varepsilon \eta^* \quad (16)$$

And finally, using the Taylor expansions (9) and (12), an approximation of this equation up to $O(\varepsilon^2)$ reads:

$$\varepsilon \frac{\partial u_1}{\partial t^*} - \frac{\varepsilon^2}{2} \frac{\partial^3 u_1}{\partial t^* \partial (x^*)^2} + \varepsilon^2 \frac{\partial u_2}{\partial t^*} + \frac{\varepsilon^2}{2} \frac{\partial (u_1^2)}{\partial x^*} + \varepsilon \frac{\partial \eta_1}{\partial x^*} + \varepsilon^2 \frac{\partial \eta_2}{\partial x^*} = \varepsilon^2 \frac{\sigma}{\rho g h^2} \frac{\partial^3 \eta_1}{\partial (x^*)^3} \cdot \varepsilon \eta^*(x^*, t^*). \quad (17)$$

Note that in this last equation, because of the Taylor-expansion performed, u_i and η_i do not depend on z .

2.3.3 Free-surface equations, lowest order

If we reduce equations (14) and (17) to their $O(\varepsilon)$ terms, we get

$$\frac{\partial \eta_1}{\partial t^*} + \frac{\partial u_1}{\partial x^*} = 0 \quad \text{and} \quad \frac{\partial u_1}{\partial t^*} + \frac{\partial \eta_1}{\partial x^*} = 0. \quad (18)$$

At this order, these two equations can be written in the form of two d'Alembert wave equations and the solutions can be decomposed into leftward and rightward components, namely:

$$\eta_1 = f(x^* - t^*) + F(x^* + t^*), \quad (19)$$

$$u_1 = f(x^* - t^*) - F(x^* + t^*). \quad (20)$$

F and f are the same for both u_1 and η_1 as equations (18) are strongly coupled. This expression shows that, to $O(\varepsilon)$, the system recovers the well-known behavior of linear non-dispersive water waves, and there is no evidence at this order that either η_1 or u_1 depends on ε . In the following section, we assume η_1 and u_1 are indeed independent of ε , and see where this approximation takes us.

2.3.4 Free-surface equations, next order

Now balancing the terms in equations (14) and (17) that are $O(\varepsilon^2)$, we get a more complex behavior:

$$\frac{\partial \eta_2}{\partial t^*} + \frac{\partial u_2}{\partial x^*} = \frac{1}{6} \frac{\partial^3 u_1}{\partial (x^*)^3} - \frac{\partial(u_1 \eta_1)}{\partial x^*}, \quad (21)$$

$$\text{and} \quad \frac{\partial u_2}{\partial t^*} + \frac{\partial \eta_2}{\partial x^*} = \left(\frac{\sigma}{\rho g h^2} - \frac{1}{2} \right) \frac{\partial^3 \eta_1}{\partial (x^*)^3} - \frac{1}{2} \frac{\partial u_1^2}{\partial x^*}. \quad (22)$$

where (18) was used to replace $\partial u_1 / \partial t^*$ by $-\partial \eta_1 / \partial x^*$.

The fact that u_1 and η_1 can be written as a combination of rightward- and leftward-traveling functions indicates that we may make the following change of variables:

$$\begin{aligned} (x^*, t^*) &\rightarrow (r, s) = (x^* - t^*, x^* + t^*), \\ \frac{\partial}{\partial x^*} &\rightarrow \frac{\partial}{\partial r} + \frac{\partial}{\partial s}, \\ \frac{\partial}{\partial t^*} &\rightarrow -\frac{\partial}{\partial r} + \frac{\partial}{\partial s}. \end{aligned}$$

so that

$$\eta_1(x^*, t^*) = f(r) + F(s) \quad (23)$$

$$u_1(x^*, t^*) = f(r) - F(s). \quad (24)$$

If we rewrite equations (21) and (22) according to the new variables r and s , we get:

$$-\frac{\partial \eta_2}{\partial r} + \frac{\partial \eta_2}{\partial s} + \frac{\partial u_2}{\partial r} + \frac{\partial u_2}{\partial s} = \frac{1}{6} \left(\frac{d^3 f}{dr^3} - \frac{d^3 F}{ds^3} \right) - \frac{df^2}{dr} + \frac{dF^2}{ds}, \quad (25)$$

$$-\frac{\partial u_2}{\partial r} + \frac{\partial u_2}{\partial s} + \frac{\partial \eta_2}{\partial r} + \frac{\partial \eta_2}{\partial s} = \left(\frac{\sigma}{\rho g h^2} - \frac{1}{2} \right) \left(\frac{d^3 f}{dr^3} + \frac{d^3 F}{ds^3} \right) - \frac{1}{2} \frac{df^2}{dr} - \frac{1}{2} \frac{dF^2}{ds} + f \frac{dF}{ds} + F \frac{df}{dr}. \quad (26)$$

Addition and subtraction of these two equations give:

$$2\frac{\partial(\eta_2 + u_2)}{\partial s} = \left(\frac{\sigma}{\rho gh^2} - \frac{1}{3}\right) \underbrace{\frac{d^3 f}{dr^3}} + \left(\frac{\sigma}{\rho gh^2} - \frac{2}{3}\right) \frac{d^3 F}{ds^3} - \frac{3}{2} \underbrace{\frac{df^2}{dr}} + \frac{1}{2} \frac{dF^2}{ds} + f \frac{dF}{ds} + F \frac{df}{dr}, \quad (27)$$

$$2\frac{\partial(\eta_2 - u_2)}{\partial r} = \left(\frac{\sigma}{\rho gh^2} - \frac{2}{3}\right) \frac{d^3 f}{dr^3} + \left(\frac{\sigma}{\rho gh^2} - \frac{1}{3}\right) \underbrace{\frac{d^3 F}{ds^3}} + \frac{1}{2} \frac{df^2}{dr} - \frac{3}{2} \underbrace{\frac{dF^2}{ds}} + f \frac{dF}{ds} + F \frac{df}{dr}. \quad (28)$$

When integrated along s and r respectively, the underbraced terms in equations (27) and (28) are multiplied by s and r respectively, which means that they diverge when $t^* \rightarrow \infty$. To be more specific, as we are talking about terms of $O(\varepsilon^2)$ compared to terms of $O(\varepsilon)$, the power series in ε no longer converges when $\varepsilon t^* \sim 1$. The other terms remain bounded as $t^* \rightarrow \infty$ and therefore do not cause any trouble.

One way to eliminate this problem would be to make sure that the underbraced terms cancel each other, leading to differential equations that do not include temporal derivative. The consequence would be that the condition should be satisfied also for the initial condition, which would be too restrictive.

2.4 Introducing the slow time-scale

In order to eliminate this problem, we need to go back to Section 2.3.3 and re-examine the assumption that u_1 and η_1 do not depend on ε . Note that if u_1 and η_1 depend on another, slowly-varying variable in addition to x^* and t^* , then derivatives with respect to that slow variable would not appear at $O(\varepsilon)$ in equation (17). We therefore introduce a slow time-scale τ , of the kind

$$\tau = \varepsilon^\alpha t^*.$$

This new variable effectively captures the possible ε dependence of η^* and \hat{u} discussed in Section 2.2, and carries over to the coefficients in their respective Taylor expansions as

$$\eta_j(x^*, t^*; \varepsilon) \rightarrow \eta_j(x^*, t^*; \tau) \quad \text{and} \quad u_j(x^*, t^*; \varepsilon) \rightarrow u_j(x^*, t^*; \tau)$$

For example, the first-order functions u_1 and η_1 are now expressed as

$$\eta_1(r, s; \tau) = f(r, \tau) + F(s, \tau) \quad (29)$$

$$u_1(r, s; \tau) = f(r, \tau) - F(s, \tau). \quad (30)$$

Finally, any temporal derivative $\partial/\partial t^*$ then becomes

$$\frac{\partial}{\partial t^*} \rightarrow \frac{\partial}{\partial t^*} + \varepsilon^\alpha \frac{\partial}{\partial \tau}.$$

All that remains to be done is to select the relevant value for α . Going back to equations (14) and (17), note that each $\partial/\partial t^*$ term results in the addition of an extra “slow” temporal derivative that is ε^α -times smaller than the “fast” derivative. Taking the simplest possible case of $\alpha = 1$, the procedure adds one extra $O(\varepsilon^2)$ term in each equation. In particular, this results in the addition of respectively $-\partial_\tau \eta_1$ and $-\partial_\tau u_1$ to the right hand sides

of equations (21) and (22). Tracking these changes further down, equations (25) and (26) acquire extra $-\partial_\tau(f + F)$ and $-\partial_\tau(f - F)$ terms in their RHS respectively, which finally leads to this alternative set of equations for u_2 and η_2 :

$$2\frac{\partial(\eta_2 + u_2)}{\partial s} = \left(\frac{\sigma}{\rho gh^2} - \frac{1}{3}\right) \frac{\partial^3 f}{\partial r^3} - \frac{3}{2} \frac{\partial f^2}{\partial r} - 2\frac{\partial f}{\partial \tau} + \mathcal{F}(r, s, \tau), \quad (31)$$

$$2\frac{\partial(\eta_2 - u_2)}{\partial r} = \left(\frac{\sigma}{\rho gh^2} - \frac{1}{3}\right) \frac{\partial^3 F}{\partial s^3} - \frac{3}{2} \frac{\partial F^2}{\partial s} + 2\frac{\partial F}{\partial \tau} + \mathcal{G}(r, s, \tau), \quad (32)$$

\mathcal{F} and \mathcal{G} being the sum of all terms in equations (27) and (28) respectively that do not cause any unphysical growth. It is now possible to eliminate all of the other terms in the RHS of each equations by requiring:

$$2\frac{\partial f}{\partial \tau} + 3f\frac{\partial f}{\partial r} + \left(\frac{1}{3} - \frac{\sigma}{\rho gh^2}\right) \frac{\partial^3 f}{\partial r^3} = 0, \quad (33)$$

$$\text{and} \quad -2\frac{\partial F}{\partial \tau} + 3F\frac{\partial F}{\partial s} + \left(\frac{1}{3} - \frac{\sigma}{\rho gh^2}\right) \frac{\partial^3 F}{\partial s^3} = 0. \quad (34)$$

These equations are two KdV equations, each one being related to either a rightward or leftward propagating wave. It is interesting to notice that both are written in the reference frame that is moving at the linear long-wave velocity (and it has been the case since the definition of s and r).

The KdV equation can be physically interpreted in the following sense. If we ignore the dispersive term in the KdV equation (*i.e.* the third derivative term), it simply reduces to the inviscid Burgers equation, which is known to generate shock waves as the nonlinear term tends to steepen the wavefronts. In the full KdV, this effect is balanced by the dispersive term, which by contrast tends to spread the horizontal extent of the fronts. As we see from the equation, these two counteracting effects act on the slow timescale only, emphasizing how important the introduction of this extra timescale was. The balance between nonlinear and dispersive effects has been ensured in Section 2.1 by the fact that $\varepsilon = O(\delta^2)$, which might a priori seem arbitrary. It is in fact a very stable feature of the system dynamics, in the sense that this balance is often eventually reached. If it is not the case, the system simply does not evolve into solitons. As a matter of fact, this balance is a property of solitons much more general than the KdV model.

3 Properties of the KdV equation

3.1 What did Korteweg and de Vries know? First properties.

It is possible to rescale equation (33) to cast it in its canonical form:

$$\frac{\partial \nu}{\partial \tau} + 6\nu\frac{\partial \nu}{\partial \xi} + \frac{\partial^3 \nu}{\partial \xi^3} = 0. \quad (35)$$

This equation supports, among others, the following spatially localized type of solutions:

$$\nu(\xi, \tau) = 2p^2 \text{sech}^2 [p(\xi - 4p^2\tau + \xi_0)], \quad (36)$$

with the hyperbolic secant defined as $\text{sech} = 1/\cosh$ and $2p^2$ the amplitude of the wave. First note the wave has a positive propagation speed $4p^2$ *in the moving reference frame* and that this speed is proportional to the amplitude of the wave. Going back to the fixed reference frame of the laboratory, it means that such waves propagate faster than the linear long waves, especially when they are of high amplitude. Such waves are called *solitons*.

When looking for non-localized solutions, it is still possible to find solutions of permanent form by introducing the Jacobi elliptical functions or cnoidal functions, which give the following set of solutions :

$$\nu(\xi, \tau) = 2p^2 \text{cn}^2 [p(\xi - 4p^2\tau + \xi_0); \kappa] + \nu_0. \quad (37)$$

When $\kappa = 0$, $\text{cn} = \cos$ and when $|\kappa| \rightarrow 1$, $\text{cn} \rightarrow \text{sech}$. In other words, when κ varies from 0 to 1, the waves transform from regular sinusoidal waves to waves with flatter troughs and sharper crests. This is in fact what happens for example as waves approach the coast, when the nonlinearity increases, as can be seen in Figure 1.



Figure 1: Waves approaching the coast of Callao (Peru). It is possible to see that as the waves approach, their crests sharpen whereas their troughs flatten (Google Earth, around $12^{\circ}01' \text{ S}$, $77^{\circ}09' \text{ W}$).

De Vries's PhD thesis ([8]) was published in 1894 and Korteweg and de Vries's article in 1895 ([13]). After having introduced their equation, having determined its periodic solutions using elliptical functions and having proved that solitons can arise out of long waves, Korteweg and de Vries stopped their work on this subject.

Interest in these nonlinear waves rose again after 1953 when Fermi, Pasta and Ulam ran one of the first numerical experiments in history, trying to know how a crystal evolves towards thermal equilibrium. In numerical experiments, they injected energy into the longest Fourier mode that fit in their periodic domain, and expected that this energy would be redistributed among all modes so that their system would tend to thermal equilibrium. Instead, they found that the energy spread into the first few modes of the system, but then returned almost entirely into the initial mode. This happened over and over again, and is now known as “Fermi-Pasta-Ulam recurrence”. This work was published in a Los Alamos technical report in 1955 ([9]) and it took 10 years until Zabusky & Kruskal (1965) ([19]) explained this phenomenon, introducing the concept (and the name) of *soliton* and reviving the KdV equation. Other names would further add their contribution to that (now) growing theory: Gardner, Miura, Greene. . . Let us bring up two of the most major properties associated with the KdV equation.

3.2 Infinite number of conservation laws

Here we only consider localized solutions of KdV, with ν infinitely differentiable and rapidly decreasing towards 0 as $|\xi| \rightarrow \infty$. A conservation law is a relation of the form

$$\frac{\partial(\text{density})}{\partial\tau} + \frac{\partial(\text{flux})}{\partial\xi} = 0.$$

Miura (1968) ([14, 15]) proved that KdV has an infinite number of conservation laws, such as:

$$\begin{aligned} \frac{\partial\nu}{\partial\tau} + \frac{\partial}{\partial\xi} \left(3\nu^2 + \frac{\partial^2\nu}{\partial\xi^2} \right) &= 0, \\ \frac{\partial\nu^2}{\partial\tau} + \frac{\partial}{\partial\xi} \left(4\nu^3 + 2\nu \frac{\partial^2\nu}{\partial\xi^2} - \frac{1}{2} \left(\frac{\partial\nu}{\partial\xi} \right)^2 \right) &= 0, \\ \frac{\partial}{\partial\tau} \left(\nu^3 - \frac{1}{2} \left(\frac{\partial\nu}{\partial\xi} \right)^2 \right) + \frac{\partial}{\partial\xi} \left(\frac{9}{2}\nu^4 + 3\nu^2 \frac{\partial^2\nu}{\partial\xi^2} - 6\nu \left(\frac{\partial\nu}{\partial\xi} \right)^2 - \frac{\partial\nu}{\partial\xi} \frac{\partial^3\nu}{\partial\xi^3} + \frac{1}{2} \left(\frac{\partial^2\nu}{\partial\xi^2} \right)^2 \right) &= 0, \\ &\vdots \end{aligned}$$

This infinite number of conservation laws makes any solution of KdV *very* constrained.

3.3 Inverse scattering transform

Useful references for this section are again Ablowitz & Segur [1] for the derivation and Dauxois & Peyrard [7] for the similarities with quantum mechanics.

Let us now take an apparently unrelated detour, and introduce the time-invariant Schrödinger equation:

$$\partial_{\xi\xi}\psi + (\lambda + \nu(\xi, \tau))\psi = 0, \tag{38}$$

where the function $\nu(\xi, \tau)$ is assumed to be infinitely differentiable and rapidly decreasing towards 0 as $|\xi|^2 \rightarrow \infty$. Note that this equation is called “time-invariant” by contrast with

the standard Schrödinger equation discussed in Lecture 3, which contains a term of the kind $i\psi_t$. As such, ν can be viewed as a localized potential which depends on the parameter τ , and as we shall see, this parameter could also be viewed as a timescale much slower than t . There are two kinds of bounded solutions to this problem:

- localized (trapped) solutions associated with discrete negative eigenvalues λ_n
- oscillatory solutions associated with positive eigenvalues $\lambda = k^2$, in the continuous spectrum

To see this, simply note that if $|\nu|$ goes to 0 rapidly as $|\xi|$ goes to infinity, then (38) reduces to $\partial_{\xi\xi}\psi = -\lambda\psi$ as $|\xi| \rightarrow \infty$. The corresponding eigenmodes ψ then either tend to $e^{\pm ik\xi}$ with $k = \sqrt{\lambda}$ (solutions which exist for all values of $\lambda > 0$), or to the exponential solutions $e^{\pm\sqrt{-\lambda}\xi}$ (in which case only a discrete number of values of λ exist for which the solution is bounded).

One way of solving this equation of course, if $\nu(\xi, \tau)$ is known, is to integrate it in space for every value of τ . Generally speaking, the eigenmodes and eigenvalues depend on τ .

Let's then ask the two following questions: (1) What conditions does the potential ν have to satisfy for the eigenvalues to be independent of τ ? Under these conditions of course, only the mode shape varies with τ which leads to the following question: (2) Can we derive a simpler evolution equation for a given mode as a function of τ , which does not rely on solving (38)?

The answer to both questions is fairly simple. Firstly, note that the only plausible evolution equation for a single mode to stay in its own phase space is

$$\partial_{\tau}\psi = A\psi + B\partial_{\xi}\psi. \quad (39)$$

(since (38) is a linear second-order equation), where A and B do not depend on ψ , but depend on ν of course. For this equation to hold as an equivalent solution of (38), a simple compatibility condition must be satisfied: $\partial_{\tau}\partial_{\xi}^2\psi = \partial_{\xi}^2\partial_{\tau}\psi$. A few manipulations reveal that this compatibility condition is equivalent to

$$\begin{aligned} \lambda_{\tau} &= -\nu_{\tau} - A_{\xi\xi} + 2B_{\xi}(\lambda + \nu) + B\nu_{\xi} \text{ and} \\ B_{\xi\xi} &= -2A_{\xi} \end{aligned} \quad (40)$$

where A and B are themselves functions of ν . For λ_{τ} to be zero, ν has to satisfy the PDE

$$\nu_{\tau} = \frac{B_{\xi\xi\xi}}{2} + 2B_{\xi}(\lambda + \nu) + B\nu_{\xi}. \quad (41)$$

In other words, any choice of the functional B yields a potential ν solution of (41) which has the desired property that the respective eigenvalues of the Schrödinger equation are independent of τ .

The relationship between the results presented above and the KdV equation becomes apparent as soon as one notes that (41) recovers KdV provided

$$B = 4\lambda - 2\nu. \text{ in which case } A = \alpha + \partial_{\xi}\nu \quad (42)$$

where α is an arbitrary constant. This fundamental point now provides a unique means to solve the KdV equation using Inverse Scattering Theory.

Indeed, note that as $|\xi| \rightarrow \infty$, equation (39) reduces to the very simple linear equation $\psi_\tau = \alpha\psi + 4\lambda\psi\xi$. Hence, if $\lim_{\xi \rightarrow \pm\infty} \psi(\xi, \tau)$ is known at time $\tau = 0$, simple linear PDE methods provide us with analytical expression for $\lim_{\xi \rightarrow \pm\infty} \psi(\xi, \tau)$ at any time τ . This information, called the “scattering data”, is *all that is needed* to reconstruct the “scattering potential” $\nu(\xi, \tau)$ at every time τ provided it is available for every eigenvalue of the system (in both the continuous and in the discrete part of the spectrum). This inversion process is called Inverse Scattering Theory.

We have therefore outlined the steps towards a novel method of computing solutions of the KdV:

1. at the initial time, given the initial condition (potential shape) $\nu(\xi, \tau = 0)$, solve equation (38) to calculate all the eigenmodes $\psi(\xi, 0)$ and associated eigenvalues λ . As it has been said previously, we ensured when choosing A and B in equation (39) that the λ_n 's are invariant in time,
2. using (39) applied as $|\xi| \rightarrow \infty$, calculate the time evolution of the scattering data.
3. invert the scattering data at any time to reconstruct the shape of the potential ν .

It can be shown that each negative eigenvalue of the Schrödinger equation represents a soliton solution of the KdV, while the continuous spectrum represents oscillatory wave trains.

Any initial condition that is bounded and smooth for all ξ , and that decreases rapidly enough as $|\xi| \rightarrow \infty$, therefore generates a KdV solution consisting of a finite number of solitons (the negative discrete eigenvalues), plus an oscillatory wave train. It is easy to construct special initial conditions that generate only solitons, and other initial conditions that generate nothing but solitons (see Section 4.1).

Finally, note that this method is sometimes referred to as a “nonlinear Fourier analysis” as these two methods are somehow similar. In the Fourier analysis of a linear problem where a solution u of a PDE associated with a dispersion relationship $\omega = \omega(k)$ has to be found, the three equivalent steps are:

1. at the initial time, calculate the Fourier transform of u :

$$\tilde{u}(k) = \int_{-\infty}^{+\infty} u(x, t = 0) e^{-ikx} dx,$$

which determines the importance of each k component. This distribution is “frozen” in time,

2. compute the evolution in time of each component. This is very easy to do as only plane waves have to be considered, the result being $\tilde{u}(k) e^{i(kx - \omega(k)t)}$,
3. reconstruct u at any time by summing all the k 's / inverting the Fourier transform:

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{u}(k) e^{i(kx - \omega(k)t)} dk.$$

4 Accuracy of the KdV equation

After deriving the KdV equation, the next natural step is to check whether it adequately models the behavior of inviscid water wave in the limit of long, weakly nonlinear waves.

4.1 KdV vs. experiment

Eight years after the rediscovery of the KdV equation by Zabusky & Kruskal (1965), Hammack (1973) [10] and Hammack & Segur (1974, 1978) [11, 12] published results comparing the solutions of the KdV equation with nonlinear wave profiles in an experimental tank. The apparatus, described in Figure 2, consists of a piston able to generate up- or downward displacement and moved by a motor that is able to generate a fairly broad range of modulations.

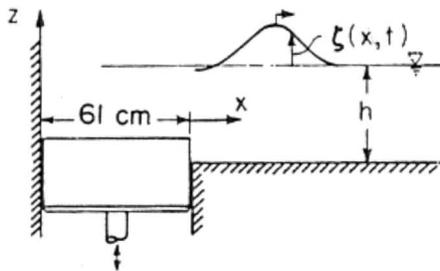


Figure 2: Description of Joe Hammack’s apparatus for the generation of localized waves: a piston is able to move up and down to generate water displacements. From [11].

Three different experiments with positive mean displacement of the piston are presented in Figure 3.

In the first experiment, the piston moves monotonically upwards. The resulting free-surface displacement is therefore step-shaped on top of it and evolves into a train of four solitons, followed by a small oscillatory tail as the wave propagates down the tank.

In the second experiment, the constant upward motion of the piston is modulated by a sine wave, which has the same period as the time it takes for the piston to move up. Close to the piston, the free-surface displacement is quite different from the previous experiments with the presence of additional “noise”, *i.e.* short, small waves superimposed on the previous step-shaped displacement. Interestingly, far away from the piston, we find again the same four solitary waves as in the first experiment.

In the third experiment, the motion of the piston is modulated with a sine wave of much higher frequency. The “noise” is initially much stronger but as the waves propagate, the “noise” propagates slower than the larger, nonlinear waves and disperses. Away from the piston, the solitary waves are recovered.

These experimental results can be understood from an analysis of the KdV equation using the IST. Solving equation (38), using the measured initial shape of the wave from the experiments (*i.e.*, at $x/h = 0$), reveals that there are exactly four negative eigenvalues, and therefore exactly four solitons in the wave form that evolved from these initial data.

This statement holds for any of the three initial conditions shown in Figure 3, and the experimental results confirmed that four solitons indeed eventually emerge.

We saw in Lecture 3 (see also equation (36)) that solitons with larger amplitudes travel faster, so it follows that the 4 solitons generated from the initial data will necessarily be ordered by amplitude for large τ , with the largest one in front. The experiments confirmed this prediction. Moreover, recall from Lecture 3 again that the KdV solutions predict a unique relationship between the amplitude and the shape of the solitons. Measurements of the amplitude of the solitons at large time, and comparison of the associated predicted profiles with the measured profiles again reveals good agreement.

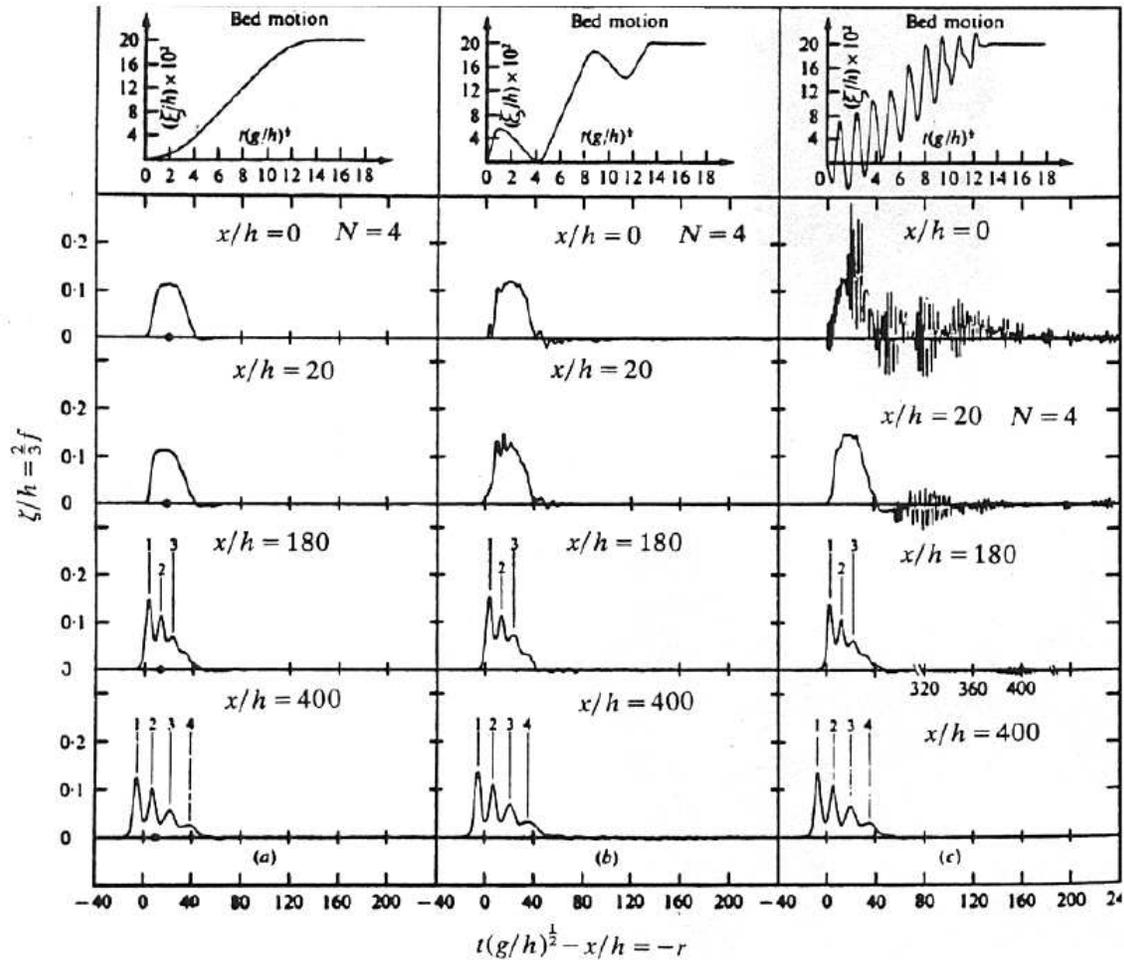


Figure 3: Various experiments resulting in the generation of KdV solitons (the general motion of the piston is positive). From top to bottom: – piston motion, – free-surface displacement on top of the piston, – free-surface displacement in three other locations further away from the piston. From left to right: three different types of piston motion. From [11].

When the piston is moved downwards ($\nu(\xi, 0) < 0$) it is possible to show that there are no discrete negative eigenvalues of (38), only modes with eigenvalues from the continuous

spectrum. As a result, we expect no generation of solitons, as can indeed be seen in Figure 4, only an oscillatory wave train. It can be seen that the amplitude is still quite large but with no implication on the appearance of solitons. This is a general result for the KdV equation: when the initial disturbance is a depression instead of an elevation, for surface waves long enough for the surface tension not to be significant, no solitary waves of depression can arise, only the continuous spectrum (cf. Section 3.3). Interestingly, if the waves are short enough for the surface tension to be significant, a surface wave of depression can correspond to an attractive potential that allows for depression solitons to exist. This corresponds to the inversion of the sign of $(\frac{1}{3} - \frac{\sigma}{\rho gh^2})$ in *e.g.* Equations (33) and (34), $\frac{\rho gh^2}{\sigma}$ being the Bond number of the problem.

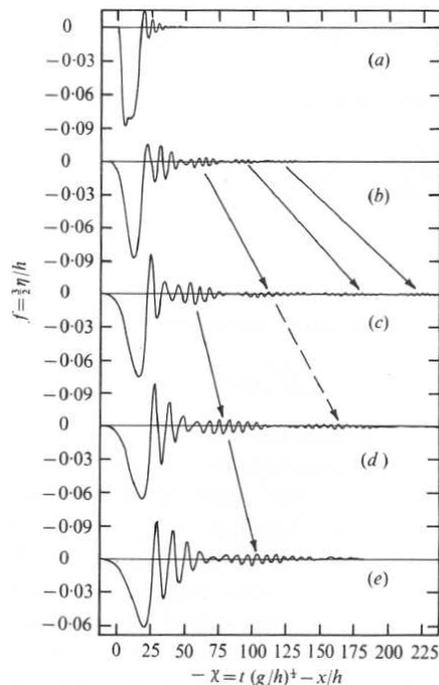


Figure 4: Experiment with a negative displacement of the piston: no solitons are generated, although the waves are large enough to produce nonlinear effects. From top to bottom: free-surface displacement in five locations with increasing distance from the piston. Arrows emphasize on the propagation of various linear wave packets. From [11].

4.2 KdV vs. full water wave equations

The question of how well the KdV equation approximates the water waves in the limit of long, nonlinear waves won't be discussed here. For more details, see [16], [20], [6, 5, 3, 4, 17, 2, 18].

5 Conclusion

This lecture was the occasion to derive the Korteweg – de Vries (KdV) equation using an asymptotic expansion of the inviscid, two-dimensional water waves in the long, weakly nonlinear limit. This expansion, restricted to the first nonlinear and dispersive terms, is derived assuming that these weak terms balance. This balance then provides a physical explanation of the fact that there exist solutions of the KdV that can propagate without any change in their shape: the solitons or solitary waves.

This equation has peculiar properties. It has an infinite number of conservation laws, making any of its solutions very constrained. It can be solved using the Inverse Scattering Transform (IST), an exact method which involves solving the equation in three linear steps instead of the original nonlinear problem. This method sorts the solutions of KdV into two different categories: solitary waves that only arise in finite numbers and a continuous spectrum of linear, dispersive waves. This method is also useful to understand that no solitary waves can emerge out of an initial depression of the surface, unless the capillary effects are significant enough.

In a series of experiments by Hammack & Segur in the 70's, the accuracy of the KdV equation for two-dimensional, long and weakly nonlinear waves has been investigated, with good qualitative agreement with the predictions of the IST and also a good quantitative agreement when dissipative correction is added.

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