Lecture 4: Zhakharov Formulation of Water Waves

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Hamiltonian mechanics is an alternative formulation of the laws of classical mechanics that was developed by William Rowan Hamilton in the early 19^{th}century. It is desirable to express mathematical systems in Hamiltonian form, as this brings us one step closer to determining whether they are completely integrable. We first present a brief overview of Hamiltonian systems, then demonstrate that the problem of inviscid surface water waves presented in the previous lectures is in fact Hamiltonian. We conclude by discussing the consequences of this result in terms of integrability.

1 Review of Hamiltonian systems

We begin with a brief introduction to Hamiltonian mechanics and the properties of Hamiltonian systems. The interested reader is referred to [1] for a more comprehensive discussion of this topic.

1.1 Formal definition

A system of $2N$ first-order ordinary differential equations is said to be Hamiltonian if there exist N pairs of coordinates in the phase space,

$$\{p_j(t), q_j(t)\}, \quad j = 1, 2, \ldots, N, \tag{1}$$

and a real-valued Hamiltonian function

$$H(\mathbf{p}(t), \mathbf{q}(t), t),$$

where $\mathbf{p}(t) = (p_1(t), \ldots, p_N(t))^T$ and $\mathbf{q}(t) = (q_1(t), \ldots, q_N(t))^T$, such that the original equations expressed in this coordinate system are:

$$\dot{q}_j = \frac{\mathrm{d}q_j}{\mathrm{d}t} = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = \frac{\mathrm{d}p_j}{\mathrm{d}t} = -\frac{\partial H}{\partial q_j}, \quad j = 1, \ldots, N. \tag{2}$$

While not all systems of $2N$ equations are Hamiltonian, many commonly known examples are. For example, consider the equation of a nonlinear oscillator,

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \omega^2\theta + \alpha\theta^3 = 0, \tag{3}$$

where $\omega, \alpha \in \mathbb{R}$. This equation describes for example the gravity-driven evolution of the angle θ of a pendulum with the downward vertical, for small angles $|\theta| \ll 1$. Indeed, for an undamped pendulum of unit length, Newton's second law states that

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -g\sin\theta = -g\left(\theta - \tfrac{1}{6}\theta^3\right) + \mathrm{O}\left(\theta^5\right), \tag{4}$$

which reduces to (3) for small angles (i.e. neglecting terms of $\mathrm{O}\left(\theta^5\right)$) and setting $\omega^2 = g$ and $\alpha = -g/6$. Choosing $q(t) = \theta(t)$ and $p(t) = \dot{\theta}(t)$, we now express (3) as the following system of first-order ordinary differential equations,

$$\dot{p} = -\omega^2 q - \alpha q^3, \quad \dot{q} = p. \tag{5}$$

We find that an appropriate choice of H is

$$H\left(p(t), q(t), t\right) = \tfrac{1}{2}p^2 + \tfrac{1}{2}\omega^2 q^2 + \tfrac{1}{4}\alpha q^4, \tag{6}$$

and it is readily verified that the Hamiltonian equations (2) yield the equations of motion (5) upon evaluation of the partial derivatives of H.

1.2 Properties of Hamiltonian systems

The Hamiltonian is frequently, but not always, the total energy of the system. In the case of the nonlinear oscillator H is exactly the constant total energy E, which may be seen by multiplying (3) by $\dot{\theta}$ and integrating,

$$E = \frac{1}{2}\dot{\theta}^2 + \frac{g}{2}\theta^2 - \frac{g}{24}\theta^4 = H(\dot{\theta}, \theta, t).$$

An essential property of a Hamiltonian system is that the 'flow' of the coordinates (1) in the phase space is volume-preserving. Consider a system of M time-dependent variables $x_j(t)$, $j = 1, \ldots, M$ governed by M first-order ordinary differential equations,

$$\frac{\mathrm{d}x_j}{\mathrm{d}t} = v_j\left(\mathbf{x}, t\right), \quad j = 1, \ldots, M. \tag{7}$$

The vector $\mathbf{x}(t) = \left(x_1(t), \ldots, x_M(t)\right)^T$ may be thought of as the coordinates of a "fluid" particle in M-dimensional phase space, and $\mathbf{v}(\mathbf{x}, t) = \left(v_1(\mathbf{x}, t), \ldots, v_M(\mathbf{x}, t)\right)^T$ as the fluid velocity vector. The fluid is "incompressible", so volume is preserved if

$$\nabla \cdot \mathbf{v} = 0. \tag{8}$$

Now suppose that $M = 2N$ and that (7) is a Hamiltonian system satisfying (2) with $p_j = x_{2j-1}$ and $q_j = x_{2j}$ for $j = 1, \ldots, N$. Then

$$\begin{aligned}\nabla \cdot \mathbf{v} = \sum_{j=1}^{M} \frac{\partial v_j}{\partial x_j} = \sum_{j=1}^{M} \frac{\partial}{\partial x_j}\left(\frac{\mathrm{d}x_j}{\mathrm{d}t}\right) &= \sum_{j=1}^{N} \frac{\partial}{\partial p_j}\left(\frac{\mathrm{d}p_j}{\mathrm{d}t}\right) + \frac{\partial}{\partial q_j}\left(\frac{\mathrm{d}q_j}{\mathrm{d}t}\right) \\ &= \sum_{j=1}^{N} \frac{\partial}{\partial p_j}\left(-\frac{\partial H}{\partial q_j}\right) + \frac{\partial}{\partial q_j}\left(\frac{\partial H}{\partial p_j}\right) = 0.\end{aligned}$$

The final equality above requires that H has continuous 2nd partial derivatives with respect to p_j and q_j. Thus every sufficiently continuous Hamiltonian system meets the "incompressibility" condition and thereby preserves volume. Note that (8) is a necessary, but not sufficient, condition for a system to be Hamiltonian. A more comprehensive discussion of this property may be found in [1], page 69.

1.3 Extension to continuous variables

We now extend Hamiltonian mechanics to systems that depend on continuous variables, such that the discrete $p_j(t)$ and $q_j(t)$ are replaced by functions $p(x,t)$ and $q(x,t)$. Consider the following nonlinear wave equation with periodic boundary conditions,

$$\theta_{tt} = c^2\theta_{xx} - \omega^2\theta - \alpha\theta^3, \tag{9}$$

where $\theta = \theta(x,t)$ and θ_t denotes partial differentiation with respect to t. We proceed as we did with the nonlinear oscillator, first seeking an energy equation. Multiplying (9) by θ_t and integrating with respect to x over the domain considered yields

$$\int \left\{\theta_t\theta_{tt} + \omega^2\theta\theta_t + \alpha\theta^3\theta_t + c^2\theta_x\theta_{xt} - c^2\left(\theta_t\theta_x\right)_x\right\} \mathrm{d}x = 0,$$

where the second derivative in x has been split to allow integration by parts. The final term in the integrand may be integrated exactly with respect to x, and disappears due to the periodic boundary conditions. The remaining terms then form an exact derivative with respect to t, which leads to the following energy equation,

$$E\left[\theta_t,\theta,t\right] = \int \left\{\tfrac{1}{2}(\theta_t)^2 + \tfrac{1}{2}c^2(\theta_x)^2 + \tfrac{1}{2}\omega^2\theta^2 + \tfrac{1}{4}\alpha\theta^4\right\} \mathrm{d}x, \qquad \frac{\mathrm{d}E}{\mathrm{d}t} = 0,$$

where E is now a functional of θ_t, θ and t. Following the method of Section 1.1, we guess that a suitable Hamiltonian is $H = E$, and choose

$$p(x,t) = \theta_t(x,t), \quad q(x,t) = \theta(x,t).$$

The new functions p and q are called the *conjugate variables*. The Hamiltonian functional of the system is then

$$H\left[p,q,t\right] = \int \left\{\tfrac{1}{2}p^2 + \tfrac{1}{2}c^2(q_x)^2 + \tfrac{1}{2}\omega^2q^2 + \tfrac{1}{4}\alpha q^4\right\} \mathrm{d}x. \tag{10}$$

If (10) is indeed the correct Hamiltonian for this system, then we expect to recover the nonlinear wave equation (9) via a set of equations analogous to (2), namely

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\delta H}{\delta p}, \quad \frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\delta H}{\delta q}, \tag{11}$$

where $\delta/\delta p$ denotes a *variational derivative* with respect to p. For a small change δp of the function p in the functional H, the variational derivative $\delta H/\delta p$ is defined as the coefficient of δp in the leading-order contribution to the integrand in the following expression:

$$H\left[p+\delta p,q,t\right] - H\left[p,q,t\right] = \int \left\{\left(\frac{\delta H}{\delta p}\right)\delta p + \mathrm{O}\left((\delta p)^2\right)\right\} \mathrm{d}x.$$

In the case of the nonlinear wave equation (10) it is straightforward to show that

$$\frac{\delta H}{\delta p} = p = \frac{\mathrm{d}q}{\mathrm{d}t},$$

but $\delta H/\delta q$ requires more work. Taking a small variation δq, we find that the integrand of δH contains a derivative of δq,

$$H[p, q + \delta q, t] - H[p, q, t] = \int \left\{ c^2 q_x (\delta q)_x + (\omega^2 q + \alpha q^3)\delta q + \mathrm{O}\left((\delta q)^2\right) \right\} \mathrm{d}x.$$

In order to express the integrand as a sum of terms proportional to powers of δq, we must integrate the first term by parts,

$$\int \left\{ c^2 q_x (\delta q)_x \right\} \mathrm{d}x = \int \left\{ c^2 (q_x \delta q)_x - c^2 q_{xx} \delta q \right\} \mathrm{d}x = -\int \left\{ c^2 q_{xx} \delta q \right\} \mathrm{d}x,$$

where the final equality follows by noting that once integrated, the boundary term vanishes. Indeed, for the perturbed functions $p + \delta p$ and $q + \delta q$ to satisfy the boundary conditions of the original problem, it is sufficient to require that the perturbations (both δq and δp) are zero on the boundary. Hence,

$$\frac{\delta H}{\delta q} = -c^2 q_{xx} + \omega^2 q + \alpha q^3 = -\frac{\mathrm{d}p}{\mathrm{d}t},$$

and so H is a Hamiltonian for the nonlinear wave equation.

2 Water waves as a Hamiltonian system

We now turn our attention to the equations of inviscid, incompressible, irrotational water waves propagating on a free fluid surface. Many attempts were made to prove that this system is Hamiltonian, but it was Vladimir Zakharov who finally published the Hamiltonian structure in 1968[2]. The full details of the following can be found, albeit in a compacted form, in Zakharov's paper.

2.1 The inviscid water wave problem

Consider an incompressible, irrotational, inviscid fluid of constant density ρ with velocity $\mathbf{u}(x, y, z, t)$ and pressure $p(x, y, z, t)$. The fluid lies above a rigid, impermeable lower boundary $z = -h(x, y)$, and has a single-valued surface at $z = \eta(x, y, t)$ that is subject to surface tension (with coefficient σ). In all that follows, we restrict our study to a finite-size domain with periodic boundary conditions, although a similar proof can be derived for an infinite domain with ϕ and η going to 0 at $\pm\infty$. This set-up is illustrated in Figure 1. In the first lecture of this series we showed that this system is governed by the following set of

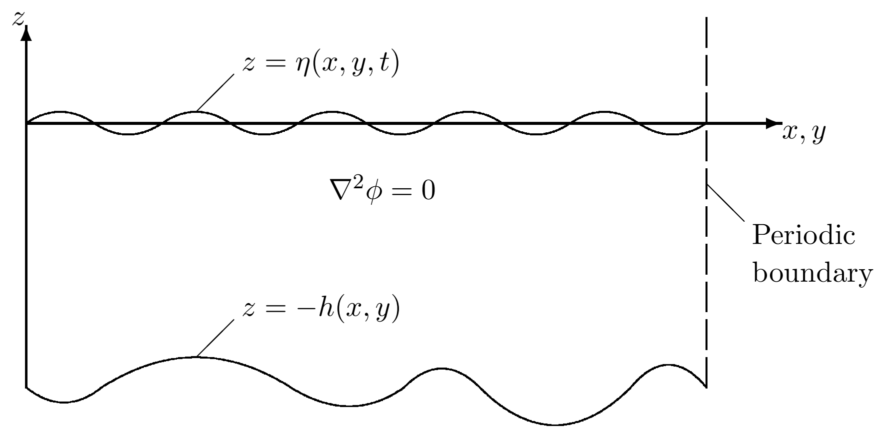

Figure 1: Capillary-gravity waves on the surface of an inviscid, incompressible, irrotational fluid of constant density.

equations and boundary conditions,

$$\frac{\partial \eta}{\partial t} + \nabla\phi \cdot \nabla\eta = \frac{\partial \phi}{\partial z} \qquad \text{on } z = \eta(x, y, t), \tag{12a}$$

$$\frac{\partial \phi}{\partial t} + \tfrac{1}{2}\left|\nabla\phi\right|^2 + g\eta = \frac{\sigma}{\rho}\nabla \cdot \left\{ \frac{\nabla\eta}{\sqrt{1 + \left|\nabla\eta\right|^2}} \right\} \qquad \text{on } z = \eta(x, y, t), \tag{12b}$$

$$\nabla^2\phi = 0 \qquad -h(x, y) < z < \eta(x, y, t), \tag{12c}$$

$$\frac{\partial \phi}{\partial z} + \nabla\phi \cdot \nabla h = 0 \qquad \text{on } z = -h(x, y), \tag{12d}$$

where $\phi(x, y, z, t)$ is the velocity potential defined as $\nabla\phi = \mathbf{u}$. The Hamiltonian structure for this system is developed entirely in terms of the surface variables $\eta(x, y, t)$ and a new function ψ,

$$\psi(x, y, t) = \phi(x, y, \eta, t),$$

the velocity potential at the free surface. This approach is plausible because although the entire system is time-dependent, the time-derivatives only appear in the boundary conditions at the free surface. That is, if at a given time $t = t_0$, $\eta(x, y, t_0)$ and $\psi(x, y, t_0)$ are known, then the solution in the rest of the fluid domain is determined uniquely by the boundary conditions and Laplace's equation (12c). The functions η and ψ are therefore plausible conjugate variables for the Hamiltonian.

We obtain an energy equation for the inviscid water wave equations by multiplying (12c) by $\partial\phi/\partial t$ and integrating over $-h(x, y) \leq z \leq \eta(x, y, t)$,

$$\int_{-h}^{\eta} \phi_t \nabla^2\phi \, \mathrm{d}z = 0.$$

Integrating by parts, we apply Leibniz's rule and the boundary conditions (12a), (12b) and (12d) to obtain the following conservation law for the energy,

$$\frac{\partial E}{\partial t} + \nabla_{H} \cdot \mathbf{F} = 0, \tag{13}$$

where

$$E(x,y,t) = \int_{-h}^{\eta} \tfrac{1}{2}|\nabla\phi|^2 \mathrm{d}z + \tfrac{1}{2}g\eta^2 + \frac{\sigma}{\rho}\left(\sqrt{1+|\nabla\eta|^2} - 1\right),$$

$$\mathbf{F}(x,y,t) = -\int_{-h}^{\eta} \phi_t \nabla_H \phi \,\mathrm{d}z - \frac{\sigma}{\rho}\frac{\eta_t \nabla\eta}{\sqrt{1+|\nabla\eta|^2}}, \tag{14}$$

are the energy flux and energy density respectively, and $\nabla_H = (\partial/\partial x, \partial/\partial y, 0)^T$ is the two-dimensional gradient vector. Note that this derivation takes a little work, and that an additional constant was added to E to set $E = 0$ if both ϕ and η are identically 0.

We obtain the expression for the total conserved energy of the system by integrating (13) over the horizontal domain $R \subset \mathbb{R}^2$,

$$\frac{\partial}{\partial t}\iint_R E \,\mathrm{d}x\,\mathrm{d}y + \int_{\partial R} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}s = 0,$$

where we have applied Green's theorem to express the second term as an integral over the boundary ∂R. Recalling that the system has periodic boundary conditions, the integral over ∂R vanishes, showing that the total energy of the system is constant:

$$\iint_R E\,\mathrm{d}x\,\mathrm{d}y = \text{constant}. \tag{15}$$

2.2 Hamiltonian structure

We now show that the inviscid water wave problem is Hamiltonian, and that the correct Hamiltonian is simply the total energy given by (15). We define

$$H[\eta,\psi] = \iint_R \mathrm{d}x\,\mathrm{d}y \left\{ \int_{-h}^{\eta} \tfrac{1}{2}|\nabla\phi|^2 \mathrm{d}z + \tfrac{1}{2}g\eta^2 + \frac{\sigma}{\rho}\left(\sqrt{1+|\nabla\eta|^2} - 1\right)\right\}, \tag{16}$$

and seek to show that the Hamilton equations,

$$\frac{\partial\eta}{\partial t} = \frac{\delta H}{\delta\psi}, \quad \frac{\partial\psi}{\partial t} = -\frac{\delta H}{\delta\eta}, \tag{17}$$

are equivalent to the boundary conditions at the free surface, (12a) and (12b).

As the Hamiltonian (16) is a functional of η and ψ, it is practical to express the boundary conditions at $z = \eta$ in terms of the surface variables. It is also desirable that their form should be similar to that of the Hamilton equations (17). We therefore rearrange (12a) and (12b) in terms of η, ψ and the component of the velocity normal to the free surface $\hat{\mathbf{n}}_\eta \cdot \nabla\phi$ (where $\hat{\mathbf{n}}_\eta$ is the vector normal to the $z = \eta$ surface). Let

$$F(x,y,z,t) = z - \eta(x,y,t),$$

such that $F(x,y,z,t) = 0$ defines the free surface. The gradient vector ∇F is then normal to the surface, so we may write the normal unit vector as

$$\hat{\mathbf{n}}_\eta = \frac{\nabla F}{|\nabla F|} = \frac{(-\eta_x, -\eta_y, 1)^T}{\sqrt{1+|\nabla\eta|^2}}.$$

The velocity normal to the surface is then

$$\nabla\phi\cdot\hat{\mathbf{n}}_\eta = \frac{-\nabla\phi\cdot\nabla\eta+\phi_z}{\sqrt{1+|\nabla\eta|^2}}, \tag{18}$$

Substituting (12a) in the above and rearranging leads to the following alternative form for the kinematic boundary condition,

$$\frac{\partial\eta}{\partial t} = \sqrt{1+|\nabla\eta|^2}\,\left[\hat{\mathbf{n}}_\eta\cdot\nabla\phi\right]_{z=\eta} \tag{19}$$

Having expressed for $\partial\eta/\partial t$ purely in terms of surface variables, we now seek a similar expression for the time derivative of ψ, our other conjugate variable. We first apply the chain rule for differentiation as follows,

$$\frac{\partial\psi}{\partial t} = \left.\frac{\partial\phi}{\partial t}\right|_{z=\eta} + \left.\frac{\partial\eta}{\partial t}\,\frac{\partial\phi}{\partial z}\right|_{z=\eta} = \left.\frac{\partial\phi}{\partial t}\right|_{z=\eta} + \left.\frac{\partial\phi}{\partial z}\right|_{z=\eta}\left[\frac{\partial\phi}{\partial z}-\nabla\phi\cdot\nabla\eta\right]_{z=\eta},$$

where the final equality follows from (12a). Substituting this into (12b) leads to the following expression for the dynamic boundary condition,

$$\begin{aligned}\frac{\partial\psi}{\partial t} + \frac{1}{2}&\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2-\left(\frac{\partial\phi}{\partial z}\right)^2\right]_{z=\eta}\\ &+\left[\frac{\partial\phi}{\partial z}\left(\nabla\phi\cdot\nabla\eta\right)\right]_{z=\eta} + g\eta - \frac{\sigma}{\rho}\nabla\cdot\left\{\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right\} = 0.\end{aligned} \tag{20}$$

Although we are unable to write $\partial\psi/\partial t$ purely in terms of surface variables, we shall see that in fact we are still able to obtain this equation from the Hamiltonian.

Our task is now to show that the Hamilton equations (17) are equivalent to the rewritten kinematic and dynamic boundary conditions at the free surface, (19) and (20). Let us first consider variations of the Hamiltonian with respect to η. Remembering that ϕ implicitly depends on η, we find that

$$\begin{aligned}H\left[\eta+\delta\eta,\psi\right]-H\left[\eta,\psi\right] = \iint_R \mathrm{d}x\mathrm{d}y\,&\left\{\int_\eta^{\eta+\delta\eta}\tfrac{1}{2}\left|\nabla(\phi+\delta\phi)\right|^2\mathrm{d}z\right.\\ &\left.+\int_{-h}^{\eta}\nabla\phi\cdot\nabla\delta\phi\,\mathrm{d}z + g\eta\,\delta\eta + \frac{\sigma}{\rho}\frac{\nabla\eta\cdot\nabla\delta\eta}{\sqrt{1+|\nabla\eta|^2}}\right\} + \mathrm{O}\left((\delta\eta)^2\right).\end{aligned} \tag{21}$$

Here we have expanded the expressions in the integrand in the small variations $\delta\eta$ and $\delta\phi$, retaining only first-order terms. We may resolve the first term in the integrand of (21) by noting that, by the definition of integration,

$$\int_\eta^{\eta+\delta\eta}\tfrac{1}{2}\left|\nabla(\phi+\delta\phi)\right|^2\mathrm{d}z = \left[\tfrac{1}{2}\left|\nabla(\phi+\delta\phi)\right|^2\right]_{z=\eta}\delta\eta+\mathrm{O}\left((\delta\eta)^2\right) = \left.\tfrac{1}{2}\left|\nabla\phi\right|^2\right|_{z=\eta}\delta\eta+\mathrm{O}\left((\delta\eta)^2\right)$$

for infinitesimally small variations $\delta\eta$. In the final term of (21) we may integrate by parts and apply Green's theorem to obtain the following,

$$\iint_R \mathrm{d}x\mathrm{d}y \int_{-h}^{\eta} \frac{\sigma}{\rho} \frac{\nabla\eta\cdot\nabla\delta\eta}{\sqrt{1+|\nabla\eta|^2}}\mathrm{d}z = \int_{\partial V} \frac{\sigma}{\rho}\frac{\delta\eta\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\cdot\hat{\mathbf{n}}\,\mathrm{d}s - \iiint_V \nabla\cdot\left\{\frac{\sigma}{\rho}\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right\}\delta\eta\,\mathrm{d}V.$$

where V denotes the volume bounded by R in the horizontal direction, and by $-h$ and η in the vertical direction. Here the first term vanishes due to the periodic boundary conditions, leaving only the second term to contribute to δH.

The second term of (21) requires more work. We note first that

$$\iiint_V \nabla\phi\cdot\nabla\delta\phi\,\mathrm{d}V = \iiint_V \nabla\cdot(\delta\phi\nabla\phi)\,\mathrm{d}V$$

because $\nabla^2\phi = 0$ everywhere in the fluid from (12c). We then apply the divergence theorem to this integral to transform it into an integral over the fluid boundaries ∂V,

$$\iiint_V \nabla\cdot(\delta\phi\,\nabla\phi)\,\mathrm{d}V = \iint_{\partial V} \delta\phi\,\nabla\phi\cdot\hat{\mathbf{n}}\,\mathrm{d}s = \iint \delta\phi\,\left[\nabla\phi\cdot\hat{\mathbf{n}}_\eta\right]_{z=\eta}\mathrm{d}s_\eta. \tag{22}$$

where $\mathrm{d}s$ denotes an infinitesimal area of the fluid boundary ∂V, while $\mathrm{d}s_\eta$ denotes an infinitesimal area on the free surface only. The final equality follows because the component of the velocity normal to the surface vanishes at $z = -h$ (12d), and because the lateral boundary terms cancel out in this periodic domain, so that in fact we are left with an integral over the free surface $z = \eta$. Noting that we may expand variations in ϕ due to variations in η at the free surface as

$$\delta\phi = \left.\frac{\partial\phi}{\partial z}\right|_{z=\eta}\delta\eta \quad \text{on } z=\eta,$$

we transform the integral on the free surface to an integral over x and y only, using the following expression for an infinitesimal area,

$$\iint \mathrm{d}s_\eta = \iint_R \sqrt{1+|\nabla\eta|^2}\,\mathrm{d}x\mathrm{d}y. \tag{23}$$

and our expression for the normal velocity at the free surface (18). We thus obtain

$$\iint \delta\phi\,\left[\hat{\mathbf{n}}_\eta\cdot\nabla\phi\right]_{z=\eta}\mathrm{d}s_\eta = \iint_R \mathrm{d}x\mathrm{d}y\left[\frac{\partial\phi}{\partial z}\delta\eta\left(\nabla\phi\cdot\nabla\eta - \frac{\partial\phi}{\partial z}\right)\right]_{z=\eta}.$$

Collecting together the above results yields the following expression for the variation of the Hamiltonian with respect to η,

$$\delta H = \iint_R \mathrm{d}x\mathrm{d}y\left\{\left[\tfrac{1}{2}|\nabla\phi|^2 + \frac{\partial\phi}{\partial z}\left(\nabla\phi\cdot\nabla\eta - \frac{\partial\phi}{\partial z}\right)\right]_{z=\eta} + g\eta - \frac{\sigma}{\rho}\nabla\cdot\left[\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right]\right\}\delta\eta. \tag{24}$$

Thus, the Hamilton equation

$$\frac{\partial \psi}{\partial t} = -\frac{\delta H}{\delta \eta}$$

is exactly equivalent to the dynamic boundary condition on the free surface (20).

Let us now consider variations of the Hamiltonian with respect to ψ. The form of (16) is such that only the first term in the integrand (the kinetic energy) depends on ψ,

$$H[\eta, \psi + \delta\psi] - H[\eta, \psi] = \delta \iiint_V \tfrac{1}{2}|\nabla\phi|^2 \mathrm{d}V.$$

However, the dependence of this term on ψ is not initially clear, so we proceed by noting that

$$\nabla\phi \cdot \nabla\phi = \nabla \cdot (\phi \nabla \phi),$$

because $\nabla^2\phi = 0$ everywhere in the fluid from (12c). We use this to rewrite our expression for the kinetic energy and apply the divergence theorem, which yields the following surface integral,

$$\iiint_V \tfrac{1}{2}|\nabla\phi|^2 \mathrm{d}z = \iint_{\partial V} \tfrac{1}{2}\phi\nabla\phi \cdot \hat{\mathbf{n}}\, \mathrm{d}s$$

where again ∂V is the surface bounding the fluid and $\mathrm{d}s$ is an infinitesimal area of that surface. By a similar argument as was used to derive equation (22), the surface integral vanishes on the horizontal and bottom boundaries, leaving only an integral over the free surface. Transforming back to an integral over the horizontal domain using (23) yields

$$\iint \tfrac{1}{2}\phi\nabla\phi \cdot \hat{\mathbf{n}}\, \mathrm{d}s_\eta = \iint_R \mathrm{d}x\, \mathrm{d}y\ \tfrac{1}{2}\psi \left[\nabla\phi \cdot \hat{\mathbf{n}}_\eta\right]_{z=\eta} \sqrt{1 + |\nabla\eta|^2}. \tag{25}$$

All that now remains is to relate $\left[\nabla\phi \cdot \hat{\mathbf{n}}_\eta\right]_{z=\eta}$ to ψ. Using a Dirichlet-to-Neumann map, it can be shown that there exists a symmetric Green's function of two variables $G(x, y; \mu, \nu)$ such that

$$\left[\nabla\phi \cdot \hat{\mathbf{n}}_\eta\right]_{z=\eta} = \iint \psi(\mu, \nu, t) G(x, y; \mu, \nu)\, \mathrm{d}s_\eta = \iint_R \mathrm{d}\mu\, \mathrm{d}\nu\ \psi(\mu, \nu, t) G(x, y; \mu, \nu) S(\mu, \nu, t), \tag{26}$$

where we have written $S(x, y, t) = \sqrt{1 + |\nabla\eta(x, y, t)|^2}$ for convenience. We may now find $\delta H/\delta\psi$ by substituting this into (25) as follows,

$$\begin{aligned} &\delta \iint_R \mathrm{d}x\, \mathrm{d}y\ \tfrac{1}{2}\psi \left[\nabla\phi \cdot \hat{\mathbf{n}}_\eta\right]_{z=\eta} S(x, y, t) \\ &= \delta \iint_R \mathrm{d}x\, \mathrm{d}y\ \tfrac{1}{2}\psi(x, y, t) S(x, y, t) \iint_R \mathrm{d}\mu\, \mathrm{d}\nu\ \psi(\mu, \nu, t) G(x, y; \mu, \nu) S(\mu, \nu, t). \end{aligned}$$

Taking variations with respect to ψ and neglecting terms of $\mathrm{O}\left((\delta\psi)^2\right)$ yields

$$\begin{aligned} \tfrac{1}{2} \iint_R \mathrm{d}x\, \mathrm{d}y\ S(x, y, t) \iint_R \mathrm{d}\mu\, \mathrm{d}\nu\ G(x, y; \mu, \nu) S(\mu, \nu, t) \\ \times \left[\psi(\mu, \nu, t)\delta\psi(x, y, t) + \psi(x, y, t)\delta\psi(\mu, \nu, t)\right] \end{aligned}$$

Finally, we use the property that the Green's function is symmetric, so we may rewrite this as,

$$\iint_R \mathrm{d}x\,\mathrm{d}y\ S(x,y,t)\delta\psi(x,y,t) \iint_R \mathrm{d}\mu\,\mathrm{d}\nu\ \psi(\mu,\nu,t)G(x,y;\mu,\nu)S(\mu,\nu,t)$$
$$= \iint_R \mathrm{d}x\,\mathrm{d}y\ \sqrt{1+|\nabla\eta|^2}\,[\nabla\phi\cdot\hat{\mathbf{n}}_\eta]_{z=\eta}\, S(x,y,t)\delta\psi,$$

where the second equality follows from (25). Thus,

$$\frac{\delta H}{\delta\psi} = \sqrt{1+|\nabla\eta|^2}\,[\nabla\phi\cdot\hat{\mathbf{n}}_\eta]_{z=\eta} = \frac{\partial\eta}{\partial t}, \tag{27}$$

as required, and so (16) is indeed the correct Hamiltonian for water waves.

3 Summary

In this lecture we have studied the foundations of Hamiltonian mechanics, and we have shown that the problem of inviscid, irrotational surface waves is Hamiltonian. It is desirable to formulate a system as a Hamiltonian because this immediately confers a number of useful properties. The fact that the "flow" in phase space is volume-preserving means that asymptotic stability of the system is impossible – only neutral stability is possible. It also means that we can not have attractors or repellers ("sources" and "sinks") in phase space. These properties make the system suitable for symplectic integration, a form of numerical integration that makes use of the preservation of volume in phase space.

An important property of a Hamiltonian formulation is that it allows us, in some cases, to identify systems of equations that are completely integrable, thus guaranteeing that the system can be solved explicitly. The following procedure is due to Liouville, and is described in more detail in [1]. We begin by defining the Poisson bracket $\{A,B\}$. For a Hamiltonian system of $2N$ ordinary differential equations (ODEs) with conjugate variables $(p_1,\ldots,p_N,q_1,\ldots,q_N)$, the Poisson bracket for any pair of functions on the phase space is

$$\{A,B\} = \sum_{j=1}^{N} \frac{\partial A}{\partial p_j}\frac{\partial B}{\partial q_j} - \frac{\partial A}{\partial q_j}\frac{\partial B}{\partial p_j}.$$

The Hamiltonian system of these 2N coupled ODEs is said to be *completely integrable* if (i) the Hamiltonian does not depend explicitly on time, and (ii) the equations admit N constants of the motion P_j (for $j = 1,\ldots,N$) that are functionally independent of each other and that are "in involution", i.e.

$$\{P_j, P_k\} = 0 \ \text{ for any pair } j,k = 1,\ldots,N.$$

Suitable combinations of these constants of motion are called "action variables". Conjugate on them are N "angle variables", Q_j (for $j = 1,\ldots,N$). The action and angle variables provide an alternative complete set of $2N$ coordinates on the phase space.

Note that while the Hamiltonian H might depend on all of the original variables $(p_1, \ldots, p_N, q_1, \ldots, q_N)$, it must now be independent of the angle variables since the action variables are constants of motion. Indeed,

$$\frac{\mathrm{d}P_j}{\mathrm{d}t} = -\frac{\partial H}{\partial Q_j} = 0, \text{ for } j = 1, \ldots, N,$$

so that

$$H = H(P_1, \ldots, P_N)$$

only, and $P_j(t)$ is constant. The other half of Hamilton's equations,

$$\frac{\mathrm{d}Q_j}{\mathrm{d}t} = \frac{\partial H}{\partial P_j}, \text{ for } j = 1, \ldots, N,$$

can be integrated trivially because $\omega_j = \partial H/\partial P_j$ is necessarily constant, so

$$Q_j(t) = \omega_j t + \phi_j, \text{ for } j = 1, \ldots, N.$$

The set of functions $P_j(t)$ and $Q_j(t)$ for $j = 1, \ldots, N$, as derived above, form the complete solution of the system for all time.

The geometry on the phase space is as follows. The N action variables define an N-dimensional manifold within the 2N-dimensional phase space. The trajectory of any solution must remain on this (time-independent) manifold for all time. If the manifold is compact, then one can show that it must be an N-dimensional torus, and that the solution must be either periodic or quasiperiodic in time. The N angle variables provide coordinates on the manifold (whether that manifold is a torus or not), and the solution consists of uniform translation along a straight line on the manifold. Thus, in terms of the action-angle variables, the solution of a completely integrable problem is very simple. However, note that the canonical transformation between the action-angle variables and the original variables $(p_1, \ldots, p_N, q_1, \ldots, q_N)$ can be complicated, so this inherently simple motion can appear quite complex when viewed in terms of the original variables.

How does this discussion of completely integrable Hamiltonian systems relate to the equations of surface water waves? As discussed above, these equations are Hamiltonian, but no one has shown that they are completely integrable (at least, not yet). However, in Lecture 5 we show that in some particular limit, the water wave problem is approximated by the Korteweg-de Vries (KdV) equation. Zakharov and Faddeev [3] took an important step in the development of soliton theory when they showed that the KdV equation can be viewed as a completely integrable, nonlinear Hamiltonian system with an infinite-dimensional phase space. In lecture 7, we discuss the Kadomtsev-Petviashvili (KP) equation, which is a natural generalization of the KdV equation; the KP equation is also completely integrable. In Lecture 13, we discuss the "three-wave equations", which approximate the water wave equations in another limit; these coupled equations are also completely integrable. In Lecture 14, we discuss the nonlinear Schrödinger equation, which approximates the water wave equations in yet another limit, and, once again, completely integrable (in one spatial dimension).

Completely integrable Hamiltonian systems are apparently rare: most Hamiltonian systems are not integrable. Before the discovery of soliton theory in the 1960s, no nontrivial examples were known of nonlinear Hamiltonian partial differential equations that are completely integrable. Now we know of infinitely many examples of such equations, but very few of them are relevant to problems of physical interest. So it is remarkable that the water wave problem has so many approximations that are completely integrable. But we live in a remarkable world.

References

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