Lecture 3: Introduction to Non-Linear Waves

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1 Introduction

The aim of this lecture is to introduce briefly the various kinds of nonlinear equations which have been proposed as models of water waves. These equations are presented here on physical grounds, and are derived more formally in following lectures. We begin by summarizing the properties of linear waves, and under which limits the linear approximations are thought to break down. Then, we investigate these limits and discuss basic nonlinear water wave equations.

1.1 Linear water waves

A one-dimensional linear wave can be represented by Fourier components

$$u = \Re \left\{ A \exp(ikx - i\omega t) \right\}, \tag{1}$$

where k is the wavenumber, ω is the frequency, and A is the amplitude. Both ω and A may be functions of k. The linear wave dynamics are determined by the dispersion relation

$$\omega = \omega(k), \tag{2}$$

the form of which depends on the circumstances. In the instance of surface water waves where surface tension is negligible, we saw in Lecture 2 that the dispersion relation is

$$\omega^2 = gk \tanh(kh) \tag{3}$$

where g is gravity and h is the still water depth. Note that there are two branches of the dispersion relation $\pm\omega$, corresponding to leftward and rightward traveling waves respectively. Depending on the physical system considered there may be any number of branches of solution. For stable waves, ω is real for all real valued k.

We also saw that there are two important velocities when considering waves,

$$c_p = \frac{\omega}{k} \tag{4}$$

$$c_g = \frac{\mathrm{d}\omega}{\mathrm{d}k}, \tag{5}$$

where c_p is the phase velocity and c_g is the group velocity. For dispersive waves, these two velocities are not the same. The phase velocity is the speed at which the phase of the wave propagates (for instance, a wave crest). Meanwhile, the wave energy propagates at the group velocity. The energy, E, of each Fourier component is usually given by the form,

$$E = F(k)|A|^2. \tag{6}$$

As will be seen in subsequent lectures, in the case of water waves the energy is given by $E = g|A|^2/2$.

As time evolves in a linear dispersive system, each Fourier component propagates at its own phase velocity and thus, a group of waves of mixed k disperses. Meanwhile, the effect of non-linearities becomes important, typically leading to three possible scenarios depending on the waves considered:

1. long waves: for $k \to 0$, the dispersion relation is only weakly dispersive as $\omega = c_0 k + O(k^3)$ (see equation 3). This dispersion is comparable with the weakly nonlinear effects in modulating the amplitude of the wave.

2. wave packets: when the wave energy is concentrated about a finite wavenumber, k_0 say, dispersion is also weak and the wave group propagates with approximately a uniform group velocity. Again, the weak dispersion is comparable with the weak nonlinearity and modulates the amplitude of the wave group.

3. resonant wave interaction: due to non-linearities, two linear waves of wavenumber k_1 and k_2 may interact to form another wave $k_0 = k_1 + k_2$. In the instance where there is resonance with $\omega_0 \approx \omega_1 + \omega_2$, this can potentially be a strong effect in amplifying and/or modulating the waves. This third scenario is discussed in other lectures.

1.2 The Korteweg-de Vries Equation

The Korteweg-de Vries (KdV) equation [6] is used to consider the weakly nonlinear, weakly dispersive behavior of the long wave case discussed in the previous section, that is, when $k \to 0$. If we use a Taylor expansion about $kh = 0$ of the dispersion relation (3), and retain the first two non-trivial terms only, we get an approximate dispersion relation with error $O(k^5)$,

$$\omega = c_0 k - \beta k^3. \tag{7}$$

where c_0 is the limit of both phase and group velocity as $k \to 0$. Identifying $-i\omega$ with $\partial/\partial t$, and ik with $\partial/\partial x$ for each Fourier component, we deduce that the evolution equation for u is:

$$u_t + c_0 u_x + \beta u_{xxx} = 0. \tag{8}$$

In the long-wave approximation the dominant terms are the first two, showing that the wave nearly propagates with constant velocity c_0 except for the weak dispersion (the third term).

The original system being nonlinear, we expect that the dispersion term is balanced by another weak, but this time nonlinear term. The exact identification of the correct nonlinear term requires a formal study of the original equations, which will be done in the following lectures. For the moment, it is sufficient to say that a term of the form $\mu u u_x$ suitably balances the dispersive term, hence introducing

$$u_t + c_0 u_x + \mu u u_x + \beta u_{xxx} = 0, \tag{9}$$

which is the KdV equation. We can simplify the equation by performing the transform $x \to x - c_0 t$, which puts the observer in a reference frame moving with velocity c_0, in which case

$$u_t + \mu u u_x + \beta u_{xxx} = 0. \tag{10}$$

Equation (10) is an integrable equation, a fact established in the 1960's. The principle solution of the KdV equation are solitons. Solitons are solitary waves, that is, they are isolated, steadily propagating pulses given by,

$$u = a\mathrm{sech}^2(\gamma(x - Vt)), \tag{11}$$

where $V = \mu a/3 = 4\beta\gamma^2$ is the soliton velocity in the moving frame.

Solitons form a one-parameter family of solutions, parameterized for example by their amplitude a. The speed, V, is proportional to the amplitude and is positive when $\beta > 0$. Conversely, V is negative when $\beta < 0$. The soliton wavenumber γ is proportional to the square root of a. As such, large-amplitude waves are thinner and travel faster. Note that solitons are waves of elevation when $\mu\beta > 0$ and of depression when $\mu\beta < 0$.

A consequence of integrability means that the initial-value problem is solvable, with methods such as the Inverse Scattering Transform (IST) for a localized initial condition (see Lecture 5). The generic outcome the initial value problem is a finite number of solitons propagating in the positive x direction and some dispersing radiation propagating in the negative x direction (when $\mu\beta > 0$).

1.3 Nonlinear Schrödinger equation

To deal with the nonlinearity associated with the wave envelopes mentioned in section 1.1 we assume that the solution is a narrow-band wave packet, where the wave energy in Fourier space is concentrated around a dominant wavenumber k_0. The dispersion relation $\omega = \omega(k)$ can then be approximated for $k \approx k_0$ by

$$\omega - \omega_0 = c_{g0}(k - k_0) + \delta(k - k_0)^2, \tag{12}$$

where $\omega_0 = \omega(k_0)$, $c_{g0} = c_g(k_0)$ and $\delta = c_{gk}(k_0)/2$, and we recall that $c_g(k) = \mathrm{d}\omega/\mathrm{d}k$, so that $c_{gk} = \omega_{kk}$. This translates to an evolution equation for the wave amplitude

$$i(A_t + c_{g0}A_x) + \delta A_{xx} = 0, \tag{13}$$

where $u = \Re[A\exp(ikx - i\omega t)]$. Here it is assumed that the envelope function $A(x,t)$ is slowly-varying with respect to the carrier phase $kx - \omega t$. The dominant term is $A_t + c_{g0}A_x \approx$

0, showing that the wave envelope propagates with the group velocity c_{g0}, modified by the effect of weak dispersion due to the term A_{xx}. This equation is well-known in quantum mechanics as the Schrödinger equation. As for the KdV equation, the small dispersion effect introduced in (13) needs to be balanced by nonlinearity. In this case, the lowest possible nonlinearity has to be a cubic term of the form $\nu|A|^2A$, for some constant ν. This form can be roughly argued on the grounds that the associated phase for all the terms in (13) should be the same, $e^{ikx-i\omega t}$. By contrast, a quadratic term for example would have phase $e^{2ikx-2i\omega t}$ or no phase at all (e.g. as in the $|A|^2$ term). A proper derivation of the NLS is presented later in this lecture series.

Thus the model evolution equation for the wave envelope is the *nonlinear* Schrödinger equation (NLS), expressed here in the reference frame moving with speed c_{g0} (as before, transform $x \to x - c_{g0}t$),

$$iA_t + \nu|A|^2A + \delta A_{xx} = 0. \tag{14}$$

Like the KdV equation, (14) is a valid model for many physical systems, including notably water waves and nonlinear optics, a result first realized in the late 1960's. Remarkably, and again like the KdV equation, it is also an integrable equation through the IST, first established by Zakharov and collaborators in 1972 ([9]). Equation (14) also has soliton solutions, and the single soliton or solitary wave solution has the form

$$A(x,t) = a\ \mathrm{sech}(\gamma(x - Vt))\ \exp(iKx - i\Omega t), \tag{15}$$

$$\gamma^2 = \frac{\nu a^2}{2\delta},\ \ V = 2\delta K,\ \text{and}\ \Omega = \delta(K^2 - \gamma^2). \tag{16}$$

This solution forms a two-parameter family, the parameters being the amplitude, a, and the "chirp" wavenumber K; however, K amounts to a perturbation of the carrier wavenumber k to $k + K$, $|K| \ll |k|$, and so can be removed by a gauge transformation. Note that this soliton solution exists only when $\delta\nu > 0$ which is the so-called focusing case.

2 Higher space dimensions

2.1 The 2D dispersion relation

In two space dimensions the wavenumber becomes a vector $\mathbf{k} = (k, l)$ and the dispersion relation is then in the form of

$$\omega = \omega(\mathbf{k}) = \omega(k, l), \tag{17}$$

where the wave phase is $\mathbf{k}\cdot\mathbf{x} - \omega t = kx + ly - \omega t$. The phase velocity is the vector $\mathbf{c} = \omega\mathbf{k}/\kappa^{\mathbf{2}}$, where $\kappa = |\mathbf{k}|$. The group velocity becomes the vector

$$\mathbf{c}_g = \nabla_{\mathbf{k}}\cdot\omega = \left(\frac{\partial\omega}{\partial k}, \frac{\partial\omega}{\partial l}\right). \tag{18}$$

Note that in general, the group velocity and the phase velocity differ in both magnitude and direction.

For water waves the dispersion relation is

$$\omega(k,l) = g\kappa \tanh \kappa h. \tag{19}$$

This is an example of an isotropic medium, where the wave frequency depends only on the wavenumber magnitude, and not its direction. In this case the group velocity is parallel to the wavenumber $\mathbf{k}$, and hence parallel to the phase velocity, with a magnitude $c_g = \mathrm{d}\omega/\mathrm{d}\kappa$.

2.2 Kadomtsev-Petviashvili equation

The Kadomtsev-Petviashvili equation (KP equation [5]) is the two-dimensional extension of the KdV equation for isotropic systems. In the reference frame moving with the linear long-wave phase speed c_0 aligned with the x-direction, the KP equation is

$$(u_t + \mu u u_x + \beta u_{xxx}) + \frac{c_0}{2} u_{yy} = 0. \tag{20}$$

Equation (20) assumes that there is weak diffraction in the y-direction, that is $\partial/\partial y \ll \partial/\partial x$. The linear terms can be deduced from the linear dispersion relation $\omega = \omega(\kappa)$, $\kappa = (k^2 + l^2)^{1/2}$, where it is assumed that $l^2 \ll k^2$. In the long-wave limit we then have $\kappa \approx k + l^2/2k$, so that

$$\omega \approx c_0 k - \beta k^3 + \frac{c_0\, l^2}{2k} \ldots . \tag{21}$$

Identifying, as in Section 1.2, $-i\omega$ with $\partial/\partial t$, $ik \sim \partial/\partial x$, and $il \sim \partial/\partial y$, we see that (20) follows. When $c_0\beta > 0$, the system is referred to as the "KPII" equation, and it can be shown that the solitary wave solution (11) is stable to transverse disturbances. This is the case for water waves. On the other hand if $\beta c_0 < 0$, (20) is the "KPI" equation for which the 1D solitary wave is unstable; instead this equation supports fully 2D "lump" solitons. Like the KdV equation, both KPI and KPII are integrable equations.

2.3 Benney-Roskes equation

Finally, for systems with an isotropic dispersion relation, the NLS equation can also be extended to two dimensions. In the reference frame moving with the group velocity c_{g0} aligned with the x-direction

$$iA_t + \nu|A^2|A + \delta A_{xx} + \delta_1 A_{yy} + QA = 0. \tag{22}$$

where Q is a functional of the amplitude $A(x,t)$. This equation deserves a little discussion. The linear term in (22) can be found by expanding the dispersion relation as in the one-dimensional case (12), so that for $k \approx k_0 \approx l \approx 0$,

$$\omega - \omega_0 = c_{g0}(k - k_0) + \delta(k - k_0)^2 + \delta_1 l^2, \tag{23}$$

where, as before $\delta = \omega_{kk}(k_0, 0)/2 = c_{gk}(k_0, 0)/2$ and $\delta_1 = \omega_{ll}(k_0, 0) = c_{g0}/2k_0$.

The final term QA on the other hand arises from the 2D extension of the cubic term. The quantity Q depends on $|A|^2$ only, a quantity which does not oscillate, and is therefore typically called the "wave-induced mean flow". The term QA is then interpreted as the effect of mean flows, generated by the nonlinear wave stresses, back on the wave-packet amplitude $A(x,t)$. The precise form of Q depends on the particular physical system being considered. For water waves, where ${c_0}^2 = gh$, it has been shown that

$$\left(1 - \frac{c_{g0}^2}{c_0^2}\right) Q_{xx} + Q_{yy} + \nu_1 |A|^2_{yy} = 0. \tag{24}$$

Note that if we set the $y-$derivatives to 0 in this equation then $Q_{xx} = 0$, and the Q term vanishes thus recovering the NLS equation.

The resulting system (22, 24) form the Benney-Roskes equations [1], also known as the Davey-Stewartson equations. Note that for water waves $\delta < 0$, $\delta_1 > 0$ and $c_{g0} < c_0$, so that (22) is hyperbolic, but (24) is elliptic.

References

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