Lecture 20: The explosive instability due to 3-wave or 4-wave mixing.

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1 Introduction

This lecture builds on the foundations of Lecture 13 where we introduced the triad interaction process. We first recall the main properties of triad interactions, before discussing one of the important cases alluded to in Lecture 13, namely that of the explosive instability.

The dispersion relation $\omega(\mathbf{k})$ for linear waves propagating in any physical system can be obtained by linearization of the governing equations (we consider a non-dissipative system here). A resonant triad exists if $\omega(\mathbf{k})$ admits 3 pairs $\{\mathbf{k}, \omega(\mathbf{k})\}$ verifying :

$$\mathbf{k}_1 \pm \mathbf{k}_2 \pm \mathbf{k}_3 = 0 \quad \text{and} \quad \omega(\mathbf{k}_1) \pm \omega(\mathbf{k}_2) \pm \omega(\mathbf{k}_3) = 0 \ . \tag{1}$$

If this is not the case, the system follows the 4-wave equations describing a resonant quartet.

1.1 3-wave equations

In the case of a triad interaction, the 3-wave amplitude equations are obtained by using the method of multiple scales. For a single triad $\{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3\}$, the solution is written as:

$$u(\mathbf{x}, t; \epsilon) = \epsilon \left[\sum_{m=1}^{3} A_m(\epsilon \mathbf{x}, \epsilon t) \exp\{i\mathbf{k}_m \cdot \mathbf{x} - i\omega_m t\} + cc \right] + O(\epsilon^2) \ , \tag{2}$$

where the amplitude depends on the slow spatial and time variables. When (2) is substituted back into the governing nonlinear equations, a compatibility condition yields the amplitude evolution equations (see Lecture 13):

$$\frac{\partial A_m}{\partial \tau} + \mathbf{c}_m \cdot \nabla A_m = i\delta_m A_n^* A_l^* \ , \tag{3}$$

with $m, n, l = \{1, 2, 3\}$ and by assumption m, n and l are all different. These three coupled equations have been studied for capillary-gravity waves or in nonlinear optics using χ_2 material. But as seen in Lecture 13, the 3-wave resonance is impossible for pure gravity waves.

1.2 4-wave equations

When 3-resonance is impossible, we are forced to consider resonant quartets with $\mathbf{k}_1 \pm \mathbf{k}_2 \pm \mathbf{k}_3 \pm \mathbf{k}_4 = 0$ and $\omega(\mathbf{k}_1) \pm \omega(\mathbf{k}_2) \pm \omega(\mathbf{k}_3) \pm \omega(\mathbf{k}_4) = 0$.

A special case always exists where :

$$\{\mathbf{k} + \delta\mathbf{k}\} + \{\mathbf{k} - \delta\mathbf{k}\} - \mathbf{k} = \mathbf{k} \ , \tag{4}$$

$$\{\omega(\mathbf{k}) + \delta\omega\} + \{\omega(\mathbf{k}) - \delta\omega\} - \omega(\mathbf{k}) = \omega(\mathbf{k}), \tag{5}$$

i.e. the resonant quartet describes the interaction between waves with very close wave numbers and frequencies. In this case, as shown in Lecture 14, the equation for the slowly varying complex amplitude naturally recovers the nonlinear Schrödinger equation ('NLS') for the one wave-field (in 1-D):

$$i\left(\frac{\partial A}{\partial \tau} + c\frac{\partial A}{\partial X}\right) + \epsilon\left[\alpha\frac{\partial^2 A}{\partial X^2} + \gamma|A|^2 A\right] = 0, \tag{6}$$

where α is a real-valued constant obtained from $\omega(k)$, and γ is the interaction coefficient and is real (for a non-dissipative problem).

For more complex interactions, with waves modes interacting with themselves and with other modes, the evolution of the amplitudes are described by coupled NLS equations. In the case of 4-wave mixing in 1-D, we have [2] :

$$i\left(\frac{\partial A_m}{\partial \tau} + c_m\frac{\partial A_m}{\partial X}\right) + \epsilon\left[\alpha_m\frac{\partial^2 A}{\partial X^2} + A_m\sum_{n=1}^{4}\gamma_{mn}|A_n|^2 + \delta_m A_p^* A_q^* A_r^*\right] = 0, \tag{7}$$

with $\{m, p, q, r\} \in \{1, 2, 3, 4\}$ and $p, q, r \neq m$, c_m is the group velocity, $\{\gamma_{mn}\}$ are interaction coefficients and $\{\delta_m\}$ are real-valued coefficients of the 4-wave mixing terms.

These NLS equations are obtained for example in the case of gravity-driven surface waves and in nonlinear optics using χ_3 material.

1.3 Explosive instability

As discussed in Lecture 13 in the case of the 3-wave equations, Coppi, Rosenbluth & Sudan in 1969 [3] showed that if $\{\delta_1, \delta_2, \delta_3\}$ in equation (3) all have the same sign, then A_1, A_2, A_3 can all blow up at the same time, everywhere in space. This is called the *explosive instability*, appropriately describing the waves blowing up everywhere to infinity in finite time (as $(\tau - \tau_o)^{-1}$). This blow-up occurs even with no spatial structure (ie $A_m = A_m(\tau)$) so there is no wave collapse or self-focussing.

B. Safdi and H.Segur [7] found in 2007 the same possibility of explosive instability even in the case of the 4-wave equations where the 3-wave resonance is impossible. Both cases are now studied in more detail.

2 Explosive instability in ODEs

2.1 3-wave mixing

As seen in Lecture 13, the simplest model of 3-wave mixing ignores the spatial dependence of the interacting modes and the set of equations described in (3) reduces to three coupled ordinary differential equations, with $\{\delta_1, \delta_2, \delta_3\}$ known and real-valued:

$$A_1'(\tau) = i\delta_1 A_2^* A_3^* \quad A_2'(\tau) = i\delta_2 A_1^* A_3^* \quad A_3'(\tau) = i\delta_3 A_1^* A_2^*. \tag{8}$$

It can be shown (see below) that if the δ_i have different sign, the solutions are bounded for all time.

Conditions for explosive instability. The following are necessary and sufficient conditions for blow-up in finite time:

- At least two of $\{A_1(0), A_2(0), A_3(0)\}$ are non-zero. Indeed, when at least two of them are zero at time $\tau = 0$ then the right-hand-sides are zero and nothing changes.
- $\{\delta_1, \delta_2, \delta_3\}$ all have the same sign and are non-zero [3].

These necessary condition are related to the existence of the constants of motion J_1, J_2 associated with the problem written in its Hamiltonian form (see Lecture 13):

$$J_1 = \frac{|A_1|^2}{\delta_1} - \frac{|A_3|^2}{\delta_3} \quad J_2 = \frac{|A_2|^2}{\delta_2} - \frac{|A_3|^2}{\delta_3}. \tag{9}$$

We see that if $\mathrm{sign}(\delta_m) \neq \mathrm{sign}(\delta_n)$ for any $m \neq n$, there is no blow-up. For instance, if $\delta_1 < 0$ and $\delta_3 > 0$, then $J_1 < 0$ and is constant, so both $|A_1|^2$ and $|A_3|^2$ are bounded and neither can blow up. But since J_2 is also constant then $|A_2|^2$ is bounded as well, so none of the wave fields can blow up. Thus it is necessary that $\{\delta_1, \delta_2, \delta_3\}$ all have the same sign and are non-zero for the amplitudes to blow-up.

Let us now suppose that $\{\delta_1, \delta_2, \delta_3\}$ all have the same sign and are non-zero. One shows by direct substitution into (8) that a special 3-parameter family of singular solutions to the governing ODEs is:

$$A_m(\tau) = \sqrt{\frac{\delta_m}{\delta_1\delta_2\delta_3}} \frac{e^{i\theta_m}}{\tau_o - \tau}, \qquad \theta_1 + \theta_2 + \theta_3 = \frac{\pi}{2} + 2\pi N \text{ , for } m = 1, 2, 3, \tag{10}$$

where $\{\tau_o, \theta_1, \theta_2\}$ are 3 free real-valued parameters. This expression readily exhibits blow-up in the finite time τ_o. To find a general solution of the ODEs, we can assume that this special solution is the first term in a Laurent series in the neighbourhood of the pole τ_o and write:

$$A_m(\tau) = \sqrt{\frac{\delta_m}{\delta_1\delta_2\delta_3}} \frac{e^{i\theta_m}}{\tau_o - \tau} \left[1 + \alpha_m(\tau_o - \tau) + \beta_m(\tau_o - \tau)^2 + \gamma_m(\tau_o - \tau)^3 + O((\tau_o - \tau)^4\right]. \tag{11}$$

By substituting this ansatz into the original ODEs, the complex coefficients can be determined order by order to yield:

$$\alpha_m = 0, \quad \Im(\beta_m) = 0, \quad \Re(\beta_1 + \beta_2 + \beta_3) = 0, \quad \Re(\gamma_m) = 0$$

$$\Im(\gamma_1) = \Im(\gamma_2) = \Im(\gamma_3) = \gamma.$$

Thus, the general solution of the equation can be written:

$$A_m(\tau) = \sqrt{\frac{\delta_m}{\delta_1\delta_2\delta_3}}\frac{e^{i\theta_m}}{\tau_o - \tau}\left[1 + \beta_m(\tau_o-\tau)^2 + i\gamma(\tau_o-\tau)^3 + O((\tau_o-\tau)^4\right] \ , \tag{12}$$

where $\{\tau_o, \theta_1, \theta_2, \beta_1, \beta_2, \gamma\}$ are 6 free real-valued constants, and $\beta_1+\beta_2+\beta_3 = 0$. Note how every nontrivial solution of the ODEs near $\tau = \tau_o$ blows up at $\tau = \tau_o$.

Mathematically, the series converges absolutely if:

1. $\beta_1 = \beta_2 = 0, |\frac{\gamma(\tau_o-\tau)^3}{3}| < 1$ or
2. $|\beta_n| \leq B, |\gamma| \leq B^{3/2}, |\tau_o - \tau|^2 B < 1.$

Physical interpretation of blow-up. Recall that the theory of non-linear resonant interactions is a weakly nonlinear theory and assumes that the 3-wave equations evolve on the long time-scale ($t = O(\epsilon^{-1})$). In finite-time blow-up, both of these model assumptions break down as $\tau \to \tau_0$, and strongly nonlinear interactions must be taken into account. For instance in the case of vorticity waves, the outcome of the explosive instability can be wave breaking or intense vortex formation [8]. Note that additional information can be obtained from the model before it breaks down completely. For example, it can reveal that significant energy is transferred into the wave modes from a background source (pump wave in nonlinear optics, shear flows in fluid dynamics, etc..). Indication to the presence of such a background source is that all four δ coefficients should have the same sign.

2.2 4-wave mixing

The equivalent 4-wave mixing equations without spatial dependence can be written:

$$iA'_m + A_m \sum_{n=1}^{4} \gamma_{\mathrm{mn}}|A_n|^2 + \delta_m A_p^* A_q^* A_r^* = 0. \tag{13}$$

These are four coupled complex ODEs with known real-valued coefficients.

The necessary and sufficient conditions for blow-up in finite time in this case are the following:

- At least three of $\{A_1(0), A_2(0), A_3(0), A_4(0)\}$ are non-zero.
- $\{\delta_1, \delta_2, \delta_3, \delta_4\}$ all have the same sign and non-zero.
- $\left|\sum_{m,n=1}^{4} \gamma_{\mathrm{mn}}|\delta_n|\right| < 4\sqrt{\delta_1\delta_2\delta_3\delta_4}.$

In the same way as in the 3-wave mixing case, it can be found [7] that the general solution of (13) is an 8-parameter family of solutions that all blow up.

Open problem. So far there is no known physical example of an explosive instability caused by 4-wave mixing. As with 3-wave mixing, an explosive instability in a 4-wave system requires a background source of energy, from which the waves modes extract energy to grow in intensity together.

2.3 Effect of damping on the blow-up

It is interesting to introduce a damping component in the governing amplitude equations to see if the blow-up can be stopped by the damping. If we add damping in (8) for example we have:

$$A'_m(\tau) = i\delta_m A_p^* A_q^* - \nu_m A_m \quad \text{with} \quad \nu_m \geq 0. \tag{14}$$

Different situations occur depending on whether the ν_m coefficients are all the same (uniform damping) or differ (non-uniform damping).

2.3.1 Uniform damping

In the simple case of uniform damping $\nu_1 = \nu_2 = \nu_3 = \nu$, it can be shown [6] that the change of variables $T = \frac{1-e^{-\nu\tau}}{\nu}$, $A_m(\tau) = e^{-\nu\tau}\alpha_m(\tau)$ (with $m = 1, 2, 3$) transforms the equation (14) into the non-damping equation (16). This shows that uniform damping does not always stop the blow-up but instead introduces a threshold for the initial amplitudes below which blow-up is suppressed. However, if $\{\delta_1, \delta_2, \delta_3\}$ all have the same sign and $|A_m(0)| \geq \sqrt{\frac{\delta_m}{\delta_1\delta_2\delta_3}} \frac{\nu}{1-e^{-\nu\tau_0}}$, then the solutions still blow-up in finite time. A similar result can be found for the 4-wave equations.

2.3.2 Non-uniform damping

In this case, the general Laurent series for the solution is too restrictive since the singularity is not a pole, so one seeks instead a solution with:

$$A_m(\tau) = \sqrt{\frac{\delta_m}{\delta_1\delta_2\delta_3}} \frac{e^{i\theta_m}}{\tau_o - \tau} \left[1 + \alpha_m(\tau_o - \tau) + b_m(\tau_o - \tau)^2 \ln(\tau_o - \tau) + \beta_m(\tau_o - \tau)^2 + ...\right] , \tag{15}$$

where the coefficients $\{\alpha_m\}$ and $\{b_m\}$ are real and depend on $\{\nu_n\}$, while the coefficients $\{\beta_m\}$ are real with $\{\beta_1, \beta_2\}$ free.

These ODEs are no longer completely integrable and the singularity is no longer a pole but blow-up persists for a 6-parameter family of solutions. The damping does not prevent blow-up, but as in the uniform case, introduces a threshold for blow-up.

3 Blow-ups in PDEs

The general set of amplitude equations for triad interactions are the following partial differential equations:

$$\frac{\partial A_m}{\partial \tau} + \mathbf{c}_m \cdot \nabla A_m = i\delta_m A_n^* A_l^* , \tag{16}$$

(for $m = 1, 2, 3$). Zakharov and Manakov [9] found that the 3-wave PDEs are completely integrable. Kaup [4] solved these equations in 1-D on whole line through numerical and analytical work to learn about blow-up. But what happens in the case of periodic boundary conditions or in more than one spatial dimension remains unknown.

Here, we use an alternative approach based on the general ODE solutions derived above:

$$A_m(\tau) = \sqrt{\frac{\delta_m}{\delta_1\delta_2\delta_3}} \frac{e^{i\theta_m}}{\tau_o - \tau} \left[1 + \beta_m(\tau_o - \tau)^2 + i\gamma(\tau_o - \tau)^3 + O((\tau_o - \tau)^4\right] , \tag{17}$$

where $\{\tau_o, \theta_1, \theta_2, \beta_1, \beta_2, \gamma\}$ are real-valued (and $\beta_1 + \beta_2 + \beta_3 = 0$). To take into account the spatial dependency, we consider a variation of the parameters, looking for $\beta_1 = \beta_1(\mathbf{x})$, $\beta_2 = \beta_2(\mathbf{x})$ and $\gamma = \gamma(\mathbf{x})$:

$$A_m(\mathbf{x}, \tau) = \sqrt{\frac{\delta_m}{\delta_1\delta_2\delta_3}} \frac{e^{i\theta_m}}{\tau_o - \tau} \left[1 + \beta_m(\mathbf{x})(\tau_o - \tau)^2 + \{g_m(\mathbf{x}) + i\gamma(\mathbf{x})\}(\tau_o - \tau)^3 + O((\tau_o - \tau)^4\right], \tag{18}$$

with $\{\beta_1(\mathbf{x}), \beta_2(\mathbf{x}), \gamma(\mathbf{x})\}$ real-valued and arbitrary while the $\{g_m(\mathbf{x})\}$ are real-valued and known (in terms of first derivatives of the $\beta_m(\mathbf{x})$ functions). Note that again we must also have $\beta_1 + \beta_2 + \beta_3 = 0$.

If we allow an additional spatial dependency for θ_1 and θ_2, then one finds that the solutions of the PDEs can be written:

$$\begin{aligned} A_m(\mathbf{x}, \tau) = \sqrt{\frac{\delta_m}{\delta_1\delta_2\delta_3}} \frac{e^{i\theta_m(\mathbf{x})}}{\tau_o - \tau} &\big[1 + ia_m(\mathbf{x})(\tau_o - \tau) \\ &+ \{\beta_m(\mathbf{x}) + ib_m(\mathbf{x})\}(\tau_o - \tau)^2 \\ &+ \{g_m(\mathbf{x}) + i\gamma(\mathbf{x})\}(\tau_o - \tau)^3 + ...\big], \end{aligned} \tag{19}$$

where the functions $a_m(\mathbf{x})$ involve first derivatives of $\{\theta_1(\mathbf{x}), \theta_2(\mathbf{x}), \theta_3(\mathbf{x})\}$, the functions $b_m(\mathbf{x})$ involve first and second derivatives of the same functions, and finally the functions $g_m(\mathbf{x})$ also involve first and second derivatives of the $\{\theta_m(\mathbf{x})\}$ as well as first derivatives of $\beta_1(\mathbf{x})$ and $\beta_2(\mathbf{x})$. Thus this family of formal solutions of the PDEs admits five free functions of $\mathbf{x}$. These functions must be infinitely differentiable, but are otherwise arbitrary, and can be members of any desired function space.

A (formal) general solution of (16), with six functions that must be real-valued and smooth, but are otherwise arbitrary, is:

$$\begin{aligned} A_m(\mathbf{x}, \tau) = \sqrt{\frac{\delta_m}{\delta_1\delta_2\delta_3}} \frac{\rho_m(\mathbf{x})e^{i\theta_m(\mathbf{x})}}{\tau_o(\mathbf{x}) - \tau} &\big[1 + ia_m(\mathbf{x})(\tau_o - \tau) \\ &+ \{\beta_m(\mathbf{x}) + ib_m(\mathbf{x})\}(\tau_o - \tau)^2 \\ &+ \{g_m(\mathbf{x}) + i\gamma(\mathbf{x})\}(\tau_o - \tau)^3 + ...\big], \end{aligned} \tag{20}$$

with the ρ_m functions verifiying:

$$\rho_1^2(\mathbf{x}) = (1 - \mathbf{c}_2 \cdot \nabla\tau_0(\mathbf{x})) \cdot (1 - \mathbf{c}_3 \cdot \nabla\tau_0(\mathbf{x})), \tag{21}$$

$$\rho_2^2(\mathbf{x}) = (1 - \mathbf{c}_3 \cdot \nabla\tau_0(\mathbf{x})) \cdot (1 - \mathbf{c}_1 \cdot \nabla\tau_0(\mathbf{x})), \tag{22}$$

$$\rho_3^2(\mathbf{x}) = (1 - \mathbf{c}_1 \cdot \nabla\tau_0(\mathbf{x})) \cdot (1 - \mathbf{c}_2 \cdot \nabla\tau_0(\mathbf{x})). \tag{23}$$

There are still open questions regarding the convergence of the series in this case, and the constraints on the free functions for blow-up.

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