Lecture 19: Wave-Mean Flow Interaction, Part II: General Theory

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1 Introduction

Suppose that we have a non-dissipative system that obeys a variational principle with a Lagrangian density, $L(\phi, \phi_t, \phi_{x_i}; t, x)$, where the field values are elements of the vector valued field $\phi(t, x_i)$, t is time and the spatial variables are x_i with $i = 1, 2, \ldots$. The governing equation of such a system is the Euler-Lagrange equation,

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \phi_t}\right) + \frac{\partial}{\partial x_i}\left(\frac{\partial L}{\partial \phi_{x_i}}\right) - \frac{\partial L}{\partial \phi} = 0\,, \tag{1}$$

where we have implied summation over the index, i. Using the calculus of variations, we can recover conservation laws which correspond to the symmetries of the Lagrangian. Energy conservation corresponds to a time symmetry, $\partial_t L = 0$, and momentum conservation corresponds to space symmetry, $\partial_{x_i} L = 0$. Wave action conservation corresponds to a phase symmetry.

If we seek a solution to describe waves superimposed on a mean flow, we introduce a phase parameter, θ, such that,

$$\phi(t, x_i, \theta + 2\pi) = \phi(t, x_i, \theta)\,. \tag{2}$$

As an example, small amplitude waves might have,

$$\phi(t, x_i) \approx a \sin(k_i x_i - \omega t + \theta)\,, \tag{3}$$

where k_i is the wavenumber vector, ω is the wave frequency and a is the wave amplitude.

Finally, we define a phase average by using the angle bracket notation,

$$\langle \cdot \rangle = \frac{1}{2\pi}\int_0^{2\pi} (\cdot) \mathrm{d}\theta\,. \tag{4}$$

Another common notation for denoting averages is the bar notation, $\bar{\cdot}$, which we will use interchangeably with the angle bracket notation, $\langle \cdot \rangle$. Note that all averaged quantities are independent of θ. Thus, we may write the entire function as a linear combination of the mean flow, $\langle \phi \rangle$, and a wave field, $\hat{\phi}$,

$$\phi = \langle \phi \rangle + \hat{\phi}\,, \tag{5}$$

where by definition, $\langle \hat{\phi} \rangle \equiv 0$.

2 Wave action

A useful consequence of the Euler-Lagrange equation (1) is that if we take any field with the same dimensions as ϕ, then we get the equation [4],

$$\frac{\partial}{\partial t}\left(\psi\frac{\partial L}{\partial \phi_t}\right)+\frac{\partial}{\partial x_i}\left(\psi\frac{\partial L}{\partial \phi_{x_i}}\right)=\psi_t\frac{\partial L}{\partial \psi_t}+\psi_{x_i}\frac{\partial L}{\partial \phi_{x_i}}+\psi\frac{\partial L}{\partial \phi}\,. \tag{6}$$

If we put $\psi=\phi_\theta$ and phase average we get,

$$\frac{\partial}{\partial t}\left\langle\hat{\phi}_\theta\frac{\partial L}{\partial \phi_t}\right\rangle+\frac{\partial}{\partial x_i}\left\langle\hat{\phi}_\theta\frac{\partial L}{\partial \phi_{x_i}}\right\rangle=0\,, \tag{7}$$

noting that the Lagrangian has no intrinsic dependence on the parameter θ. If we define the following quantities,

$$A=\langle\hat{\phi}_\theta\frac{\partial L}{\partial \phi_t}\rangle,\quad B_i=\langle\hat{\phi}_\theta\frac{\partial L}{\partial \phi_{x_i}}\rangle\,, \tag{8}$$

we can see that the corresponding phase symmetry is,

$$\frac{\partial A}{\partial t}+\frac{\partial B_i}{\partial x_i}=0\,. \tag{9}$$

A is the **wave action density** and B_i is the **wave action flux**. Physically, the wave action is the wave energy divided by the intrinsic wave frequency (which we shall define later). So, when a wave's frequency decreases (increases), the wave gains (loses) energy from (to) the mean flow in order to conserve wave action. Being wave quantities, if A and $\boldsymbol{B}$ are zero, then there are no waves. As such, wave action is a good measure of wave activity. Equation (9) is a conservation law in all unforced, non-dissipative systems (as is the case with the system we are considering). Formally, the wave action conservation law is valid without restriction on amplitude or on the relative time and space scales of the waves (with respect to the mean flow).

If we now suppose that the mean flow, background medium and wave parameters are slowly varying, we may write the wave field as,

$$\hat{\phi}\sim\hat{\phi}(S(t,x_i)+\theta;t,x_i)\,. \tag{10}$$

As we are assuming that S varies much more rapidly than the wave field's explicit dependence on t or x_i, the explicit derivatives $\partial\hat{\phi}/\partial x_i$ and $\partial\hat{\phi}/\partial t$ are small terms. Therefore, we can write

$$\hat{\phi}_t\sim\hat{\phi}_\theta\frac{\partial S}{\partial t}\quad\hat{\phi}_{x_i}\sim\hat{\phi}_\theta\frac{\partial S}{\partial x_i}\,, \tag{11}$$

noting that,

$$\frac{\partial S}{\partial t}=-\omega\quad\frac{\partial S}{\partial x_i}=\kappa_i\,, \tag{12}$$

where ω is the local frequency and κ_i is the local wavenumber. Substituting equations (12) into equations (11), we get

$$\hat{\phi}_t\sim-\omega\hat{\phi}_\theta\quad\hat{\phi}_{x_i}\sim\kappa_{x_i}\hat{\phi}_\theta\,. \tag{13}$$

Substitution into equations (8) yields

$$A = \left\langle \hat{\phi}_\theta \frac{\partial L}{\partial(-\omega\hat{\phi}_\theta)} \right\rangle \quad B = \left\langle \hat{\phi}_\theta \frac{\partial L}{\partial(\kappa_i\hat{\phi}_\theta)} \right\rangle . \tag{14}$$

Both A and B are $O(a^2)$, where a is the wave amplitude in the small amplitude limit, although, these equations are formally valid without any restriction on amplitude. It can be shown that

$$\left\langle \hat{\phi}_\theta \frac{\partial L}{\partial(-\omega\hat{\phi}_\theta)} \right\rangle = \hat{\phi}_\theta \hat{\phi}_\theta^{-1} \frac{\partial \langle L \rangle}{\partial(-\omega)}$$
$$\Rightarrow A = -\frac{\partial \bar{L}}{\partial \omega} , \tag{15}$$

where we note that ω is already an averaged quantity. Similarly for B_i, we get,

$$B_i = \frac{\partial \bar{L}}{\partial \kappa_i} . \tag{16}$$

where again we note that κ_i is an averaged quantity. Here $\bar{L} \equiv \langle L \rangle$, is the averaged Lagrangian and, similarly to ω and κ_i, is a slowly varying function of t and x_i.

Finally for slowly varying waves, we also require the equation for conservation of wave action,

$$\frac{\partial \kappa_i}{\partial t} + \frac{\partial \omega}{\partial x_i} = 0 \tag{17}$$

3 Linearised waves

We are now in a position where we can decompose the Lagrangian as follows,

$$L = L_0(\bar{\phi}_t, \bar{\phi}_{x_i}, \bar{\phi}; t, x_i) + L_1(\hat{\phi}_t, \hat{\phi}_{x_i}, \hat{\phi}; t, x_i) , \tag{18}$$

where $\bar{\phi} \equiv \langle \phi \rangle$. Here $L_0 = L(\bar{\phi}, \bar{\phi}_t, \bar{\phi}_{x_i}; t, x_i)$ is the Lagrangian for the mean flow and $L_1 = L(\hat{\phi}, \hat{\phi}_t, \hat{\phi}_{x_i}; tx_i)$ is the Lagrangian for the wave field. While the Lagrangian for the wave field depends on the Lagrangian for the mean flow, the dependence is not explicit but instead is present implicitly via the explicit dependence on t and x_i.

We may further simplify equation (10) by assuming a small amplitude,

$$\hat{\phi}(t, x_i) \approx a(t, x_i) \sin(S(t, x_i) + \theta) . \tag{19}$$

If we substitute into equation (18) and then average, we find that the average of the Lagrangian wave field is, to leading order, given by

$$\bar{L}_1 \approx D(\omega^*, \kappa_i; t, x_i) a^2 . \tag{20}$$

We note that there are no linear terms in a, as these terms average to zero, furthermore, the error is $O(a^4)$. In equation (20), we have extracted the dependence of the wave field on the mean field, where U_i is the **mean velocity field** and ω^* is the **intrinsic frequency** given by $\omega^* = \omega - U_i\kappa_i$, while the rest of the fields remain suppressed in the explicit x_i and t dependencies. In the linearised approximation, the mean fields are known, with the result that $\bar{L}_1$ depends only on the mean velocity U_i through the intrinsic frequency, ω^*, which follows from Galilean invariance.

If we substitute equation (20) into our expressions for A and B, equations (15) and (16), we obtain,

$$A = -\frac{\partial D}{\partial \omega^*}a^2 \tag{21}$$

$$B = \frac{\partial D}{\partial \kappa_i}a^2\,, \tag{22}$$

where we note that $(\partial D/\partial\omega^*)(\partial\omega^*/\partial\omega) = \partial D/\partial\omega^*$ as U_i is slowly varying. Furthermore, we note that the variation of $\bar{L}$ is independent of a, $\partial\bar{L}_1/\partial a = 0$. As a result of this latter relation, we find that the dispersion relation is,

$$\begin{aligned} &D(\omega^*, \kappa_i; t, x_i)2a = 0 \\ &\therefore D(\omega^*, \kappa_i; t, x_i) = 0\,. \end{aligned} \tag{23}$$

It follows, upon differentiation with respect to κ_i (remembering the definition of the intrinsic frequency) that,

$$\frac{\partial D}{\partial \omega^*}\boldsymbol{c}_{g_i} + \frac{\partial D}{\partial \kappa_i} = 0\,, \tag{24}$$

where $\boldsymbol{c}_{g_i} = \partial\omega/\partial\kappa_i = U_i + \boldsymbol{c}_{g_i}^*$ is the **group velocity** and $\boldsymbol{c}_{g_i}^* = \partial\omega^*/\partial\kappa_i$ is the intrinsic group velocity. Use of equations (21) and (22) gives us the relation $B_i = \boldsymbol{c}_{g_i}A$, which, upon substitution into the wave action equation (9), yields the result,

$$\frac{\partial A}{\partial t} + \frac{\partial(\boldsymbol{c}_{g_i}A)}{\partial x_i} = 0\,. \tag{25}$$

4 Energy and momentum

In order to provide a more physical interpretation of wave action, we consider the conservation laws for energy and momentum. Supposing that we have a full Lagrangian system governed by equation (1), $L(\phi, \phi_{x_s}; x_s)$ where $s = 0, 1, 2, 3$ and we note that $x_0 = t$ is a time-like variable. Recalling equation (6), we write,

$$\frac{\partial}{\partial x_s}\left(\psi\frac{\partial L}{\partial \phi_{x_s}}\right) = \psi_{x_s}\frac{\partial L}{\partial \phi_{x_r}} + \psi\frac{\partial L}{\partial \phi}\,.$$

If we now put $\psi = \phi_{x_r}$, we find a conservation law,

$$\begin{aligned} \frac{\partial}{\partial x_s}\left(\phi_{x_r}\frac{\partial L}{\partial \phi_{x_s}}\right) &= \frac{\partial \phi_{x_r}}{\partial x_s}\frac{\partial L}{\partial \phi_{x_r}} + \frac{\partial \phi}{\partial x_r}\frac{\partial L}{\partial \phi} \\ \frac{\partial}{\partial x_s}\left(\phi_{x_r}\frac{\partial L}{\partial \phi_{x_s}}\right) &= \frac{\mathrm{d}L}{\mathrm{d}x_r} - \left(\frac{\partial L}{\partial x_r}\right)_e\,, \end{aligned} \tag{26}$$

where $(\cdot)_e$ indicates that the derivative is taken keeping ϕ and ϕ_{x_r} fixed while,

$$\frac{\mathrm{d}L}{\mathrm{d}x_r} \equiv \frac{\partial L}{\partial x_r} + \phi_{x_r}\frac{\partial L}{\partial \phi} + \frac{\partial L}{\partial \phi_{x_s}}\frac{\partial \phi_{x_s}}{\partial \phi_{x_r}} \tag{27}$$

may be recognised as the total derivative. Furthermore, we note that $\frac{\mathrm{d}L}{\mathrm{d}x_r} = \delta_{rs}\frac{\mathrm{d}L}{\mathrm{d}x_r} = \frac{\partial}{\partial x_r}\delta_{rs}L$ in which δ_{rs} is the Kronecker delta symbol. This gives us the result

$$\begin{aligned} \frac{\partial}{\partial x_s}\left(\phi_{x_r}\frac{\partial L}{\partial \phi_{x_s}} - L\delta_{rs}\right) &= \frac{\partial L}{\partial x_r} \\ \frac{\partial T_{rs}}{\partial x_s} &= -\frac{\partial L}{\partial x_r}\,, \end{aligned} \tag{28}$$

where $T_{rs} = \phi_{x_r}\frac{\partial L}{\partial \phi_{x_s}} - L\delta_{rs}$ can be identified as the energy-momentum tensor from classical physics.

While the exact components of the tensor depend on the problem being studied, we can identify T_{00} as the energy density, T_{0j} as the energy flux and T_{i0} as the momentum density and T_{ij} as the corresponding fluxes. We may apply the averaging operator to the conservation law, equation (28), which gives us the averaged total energy $\langle T_{00}\rangle$ and averaged total momentum, $\langle T_{i0}\rangle$. These are not, on their own, particularly useful, as they contain both the mean and wave fields. Typically, the wave field is $O(a^2)$, and as such, is small relative to the mean field contribution.

We follow the averaging procedure of [2], by putting $\psi = \hat{\phi}_{x_s}$, recalling that $\phi = \bar{\phi} + \hat{\phi}$, and essentially follow the same procedure as for the full wave field to yield,

$$\frac{\partial T_{rs}}{\partial x_s} = -\frac{\partial \bar{L}_1}{\partial x_r} \tag{29}$$

where now

$$T_{rs} = \left\langle \hat{\phi}_{x_r}\frac{\partial L_1}{\partial \hat{\phi}_{x_s}} - \delta_{rs}L_1 \right\rangle . \tag{30}$$

We may now identify T_{00} as the pseudoenergy density, T_{0j} as the pseudoenergy flux and T_{i0} as the pseudomomentum density and T_{ij} is the pseudomomentum flux. Recalling that $\phi = \bar{\phi} + \hat{\phi}$, we can see that we are able to replace ϕ with $\hat{\phi}$ throughout as there is a linear relation between the two, with the value for $\bar{\phi}$ and its derivatives being slowly varying. We also recall that the total Lagrangian is a linear combination of the average and wave Lagrangians, $L = L_0 + L_1$. However, we note that L_0 is not a function of $\hat{\phi}$, and as such terms like $\partial L_0/\partial\hat{\phi} = 0$. These approximations make T_{rs} an $O(a^2)$ wave property (where we recall that a is the wave amplitude). Note, however, that equation (30) is not a conservation law unless L_1 is independent of x_s. As a consequence, the mean flow, $\bar{\phi}$ is also required to be independent of x_s.

If we put $\psi = \hat{\phi}_\theta$, we regain equation (8). If a suitable ergodic principle exists, we may identify the phase with a particular coordinate, $\theta = x_s$. This allows us to identify $T_{s0} = A$ and $T_{si} = B_i$. We note that the diagonal term T_{ss} is thus absent from the conservation law, equation (30). As such, we can now say that wave action is pseudoenergy for time averaging and is pseudomomentum for space averaging.

Generally speaking, wave energy is not as useful a quantity as wave action, since wave energy is not generally conserved. If we suppose that the mean flow consists of a mean velocity U_i, and a vector valued mean field, λ, that we require to satisfy the equation,

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} + \Lambda_{ij}\lambda\frac{\partial U_i}{\partial x_j} = 0\,, \tag{31}$$

where $\frac{\mathrm{d}}{\mathrm{d}t} \equiv \frac{\partial}{\partial t} + U_i\frac{\partial}{\partial x_i}$ is the material derivative and $\mathbf{\Lambda}$ is a dyadic of basis vectors. Physically, λ incorporates quantities such as mean depth, mean density or even mean magnetic field. We can now follow [3] and define the wave energy E as the pseudoenergy in a reference frame moving with the mean flow,

$$E = T_{00} + U_iT_{i0} \tag{32}$$

$$E = \left\langle \frac{\mathrm{d}\hat{\phi}}{\mathrm{d}t}\frac{\partial L_1}{\partial\hat{\phi}_t} - L_1 \right\rangle\,, \tag{33}$$

where we note that $T_{00} = \langle\frac{\partial\hat{\phi}}{\partial t}\frac{\partial L_1}{\partial\hat{\phi}_t} - L_1\rangle$ and $U_iT_{i0} = \langle U_i\hat{\phi}_{x_i}\frac{\partial L_1}{\partial\hat{\phi}_t}\rangle$ from equation (30). Similarly, the wave energy flux is given by,

$$F_i = \left\langle \frac{\mathrm{d}\hat{\phi}}{\mathrm{d}t}\frac{\partial L_1}{\partial\hat{\phi}_{x_i}} - U_iL_1 \right\rangle\,. \tag{34}$$

Here, we note that $L_1 = L_1(\hat{\phi}, \hat{\phi}_t, \hat{\phi}_{x_i}; U_i, \lambda; x_i, t)$. If we suppose that the dependence of E and $\boldsymbol{F}$ is solely through the material derivative, then it can be shown that

$$\frac{\partial E}{\partial t} + \frac{\partial F_i}{\partial x_i} = -R_{ij}\frac{\partial U_i}{\partial x_j} - \left(\frac{\mathrm{d}\bar{L}_1}{\mathrm{d}t}\right)_e \tag{35}$$

where $(\ldots)_e$ indicates the explicit derivative with respect to t and x_i while the wave field, $\hat{\phi}$ and the mean fields, U_i, λ are held constant and $\boldsymbol{R}$ is the **radiation stress tensor**,

$$R_{ij} = -T_{ij} + U_jT_{i0} - \Lambda_{ij}\lambda\frac{\partial\bar{L}_1}{\partial\lambda}\,. \tag{36}$$

The final piece of the puzzle is an equation for the mean flow, which is gained by variation of the mean field. In order to achieve this, we apply the averaging operator, equation (4), to the Lagrangian, equation (1), subject to the constraint described by equation (31),

$$\frac{\partial}{\partial t}\left(\frac{\partial L_0}{\partial U_i}\right) + \frac{\partial}{\partial x_j}\left(U_i\frac{\partial L_0}{\partial U_i} - \Lambda_{ij}\lambda\frac{\partial L_0}{\partial\lambda} + L_0\delta_{ij}\right) - \left(\frac{\partial L_0}{\partial x_i}\right)_e = -\frac{\partial R_ij}{\partial x_j} + \left(\frac{\partial\bar{L}_1}{\partial x_i}\right)_e\,. \tag{37}$$

This equation may also be derived using Whitham's variational principle. When $\mathbf{\Lambda}$ is isotropic, $\Lambda_{ij} = M\delta_{ij}$, then there is a mean pressure Q, such that equation (37) becomes,

$$-\frac{\partial R_{ij}}{\partial x_j} = -\frac{\partial T_{i0}}{\partial t} - \frac{\partial(U_jT_{i0})}{\partial x_j} + \frac{\partial Q}{\partial x_i}\,. \tag{38}$$

5 Slowly varying waves

In the circumstances where we have a slowly varying, almost-plane wave field (where the dependence on phase $S(x_i)$ is rapidly varying compared to the explicit dependence of the field on x_i), which can be reasonably approximated by equation (10). Recall that under such circumstances the frequency is given by $\omega = -\partial S/\partial t$ and wavenumber is given by $\kappa_i = \partial S/\partial x_i$. We also get useful reductions for pseudoenergy,

$$T_{00} \approx \omega A - \bar{L}_1 \tag{39}$$

$$T_{0i} \approx \omega B_i \,, \tag{40}$$

for pseudomomentum,

$$T_{i0} \approx -\kappa_i A \tag{41}$$

$$T_{ij} \approx -\kappa_i B_j - \bar{L}_1 \delta_{ij} \,, \tag{42}$$

for wave energy,

$$E \approx \omega^* A - \bar{L}_1 \tag{43}$$

$$F \approx \omega^*(B_i - U_i A) \,, \tag{44}$$

and the radiation stress tensor reduces to

$$R_{ij} \approx \kappa_i (B_j - U_j A) + \bar{L}_1 \delta_{ij} - \Lambda_{ij} \lambda \frac{\partial \bar{L}_1}{\partial \lambda} \,. \tag{45}$$

For linearised waves, we can achieve further results by recalling the dispersion relation, equation (23), which implies that the intrinsic frequency is $\omega^* = \Omega(\kappa_i; \lambda; x_i, t)$, and $\omega = \kappa_i U_i + \omega^*$. We also recall the wave action equation, equation (25) with the group velocity given by $\boldsymbol{c}_{g_i} = U_i + \partial\Omega/\partial\kappa_i$. Noting, that for linearised waves $\bar{L}_1 = 0$, which implies that the wave energy is given by $E = \omega^* A$ and the pseudoenergy is give by $T_{00} = \omega A$, giving new expressions for the wave energy equation (35) and the radiation stress equation (36),

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_i}([U_i + \boldsymbol{c}_{g_i}^*]E) = -R_{ij}\frac{\partial U_i}{\partial x_j} + \frac{E}{\omega^*}\left(\frac{\mathrm{d}\Omega}{\mathrm{d}t}\right)_e \tag{46}$$

$$R_{ij} = A\left(\kappa_i \boldsymbol{c}_{g_j} + \Lambda_{ij}\lambda\frac{\partial \Omega}{\partial \lambda}\right) \,. \tag{47}$$

It is worth noting that there are some linear, almost plane waves for which this approximation does not hold, with one of the more notable exceptions being Rossby Waves.

6 Extensions

The theory outlined above can be extended to modal waves. That is, waves that are confined to a wave guide, such that the direction of propagation is in a reduced set of spatial dimensions and a modal structure in the remaining set of dimensions. Examples of such waves are water waves and internal waves.

The general theory outlined above can still be used, although we need to combine the Lagrangian averaging with integration across the waveguide.

In order to apply the theory outlined in the previous sections to fluid flows, we need to identify a suitable Lagrangian. Sometimes it is possible to find it directly from the Eulerian formulation, however, it is usually most convenient to find the Lagrangian using a Lagrangian formulation of the equations of motion.

In order to consider finite amplitude waves, it is best to use **generalised Lagrangian mean theory** (GLM) developed by [1, 2]. In GLM theory, we define the particle displacements from a mean position that moves with the mean velocity U_i. As such, x_i are Lagrangian variables moving with the Lagrangian mean velocity U_i, relative to which the particle displacements are defined as ξ_i. The Eulerian variables thus become $x_i' = x_i + \xi_i$ and the Eulerian velocity becomes

$$u_i' = U_i + \frac{\mathrm{d}\xi_i}{\mathrm{d}t}\,, \tag{48}$$

where we note that the material derivative is $\frac{\mathrm{d}}{\mathrm{d}t} \equiv \frac{\partial}{\partial t} + U_i \frac{\partial}{\partial x_i}$, and we note that $\langle \xi_i \rangle = 0$.

References

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