Lecture 13: Triad (or 3-wave) resonances.

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1 Introduction

The previous lectures have been dedicated to the study of specific nonlinear problems, most of them involving solitons. It is now time to discuss the problem of triad resonances, as they are actually the most basic nonlinear phenomenon that can occur between two waves. As their name indicates, triad resonances involve the weak interaction between two waves combining to form a third wave.

We construct a set of equations describing single-triad interactions in Section 2, and study their mathematical structure in Section 3. Insights into the behavior of a real system, subject to multiple triad resonances, are presented in Sections 4 and 5.

2 Derivation of the 3-wave equations

2.1 Weakly nonlinear interactions

For dispersive waves of small amplitude, resonant triad interactions are the first nonlinear interactions to appear (if they are possible). Let's start with a physical system of evolutionary partial differential equations with no dissipation, described by a set of equations

$$N(u) = 0$$

with $u = 0$ being the state of rest.

To study the nonlinear interaction of three waves of weak amplitude, the procedure is, as usual, to linearize around $u = 0$. From previous lectures, we expect the linear solution to take the form:

$$u(\boldsymbol{x}, t, \varepsilon) = \varepsilon \left[\sum_k A(\boldsymbol{k}) e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega(\boldsymbol{k})t)} + c.c. \right] + O(\varepsilon^2)$$

This linear problem is associated with a dispersion relation:

$$\omega = \omega(\boldsymbol{k}).$$

Once this dispersion relation is found, the next step is to build a weakly nonlinear model. In order to do so, one has to see if the linear problem admits three pairs $\{\boldsymbol{k}_i, \omega(\boldsymbol{k}_i)\}$, with $\boldsymbol{k}_i \neq 0$ for $i = 1, 2, 3$, satisfying a triad relationship:

$$\boldsymbol{k_1} \pm \boldsymbol{k_2} \pm \boldsymbol{k_3} = 0, \tag{1}$$

$$\omega(\boldsymbol{k_1}) \pm \omega(\boldsymbol{k_2}) \pm \omega(\boldsymbol{k_3}) = 0. \tag{2}$$

A graphical procedure to investigate this possibility has been discovered several times (*e.g.* Ziman (1960) [18], Ball (1964) [1]) and is illustrated in Figure 1. In Figure 1(a), the solid line represents the linearized dispersion relation of pure gravity waves in deep water:

$$\omega = \pm\sqrt{g|k|},$$

while Figure 1(b) shows the linearized dispersion relation of capillary-gravity waves in deep water:

$$\omega = \pm\sqrt{g|k| + \frac{\sigma}{\rho}|k|^3}.$$

The graphical procedure to detect resonant triads is the following. First, pick any point (k_1, ω_1) on one of the branches (solid lines) of the dispersion curve. Then reproduce all branches of the dispersion relation with the origin translated to (k_1, ω_1), here drawn as the dashed lines. Let P be a point where the two curves intersect. Its coordinates in the original coordinate system are identified as (k_3, ω_3), and as (k_2, ω_2) in the translated coordinate system. Then by construction,

$$k_1 + k_2 = k_3,$$
$$\omega_1 + \omega_2 = \omega_3.$$

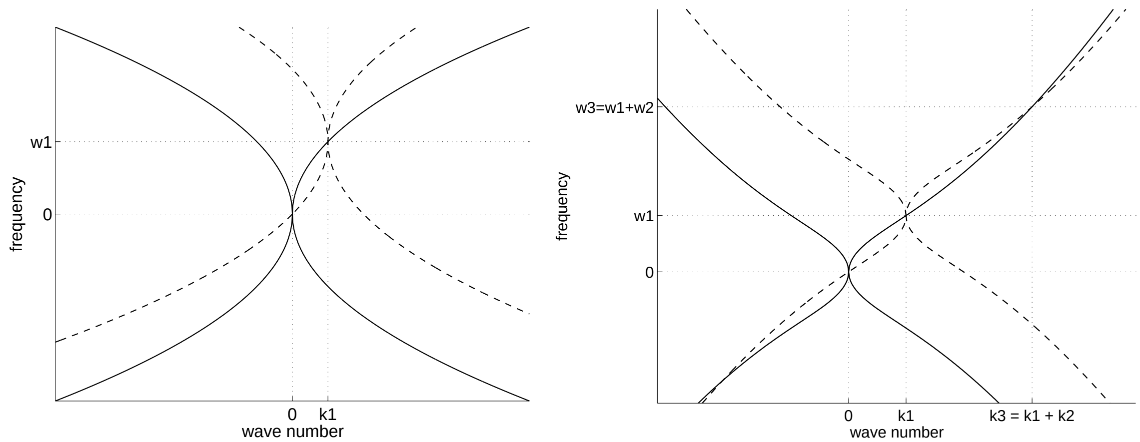
In Figure 1(a), the only point of intersection is at the origin on the solid curve ($k_3 = 0$) contrary to our requirements. Hence, there is no possibility of forming any resonant triad with pure gravity waves. In Figure 1(b) on the other hand one can notice that there are at least two possibilities of forming a resonant triad in top right and in the bottom left hand corner of the figure. In fact, for most choices of k_1 there will be at least two intersection points, leading to another two new triads. Hence, for capillary-gravity waves in 1D, there are infinitely many possible triads.

We have just described the 1-D case in which k_1, k_2 and k_3 are collinear, so they are effectively scalars. In 2D, $\boldsymbol{k_1}$, $\boldsymbol{k_2}$ and $\boldsymbol{k_3}$ are two-component vectors. A similar procedure can be applied, except that both solid and dashed curves now become 2-D surfaces. The two surfaces typically intersect on 1-D curves (or not at all), showing that in 2D there are an infinite number of possible triads associated with a given pair $(\boldsymbol{k_1}, \omega(\boldsymbol{k_1}))$.

To conclude, there are two simple alternatives: if triplets of waves satisfying equations (1) and (2) exist, then triad resonances are expected. If there are no such triplets, one has to look for resonant quartets (4-waves interactions). The latter alternative is discussed in Lectures 14, 15 and 20.

2.2 Single triad

Suppose that equations (1) and (2) are satisfied for exactly one triad. Let us write u as a superposition of the three interacting waves plus weak interaction terms:



(a) Gravity waves, deep water

(b) Capillary-gravity waves, deep water

Figure 1: Linearized dispersion relations of: – (a) pure gravity waves in deep water, – (b) capillary-gravity waves in deep water. In (a), no resonant triad can be formed and in (b), (k_1, ω_1), (k_2, ω_2) and (k_3, ω_3) form a resonant triad.

$$u(\boldsymbol{x}, t; \varepsilon) = \varepsilon \left[\sum_{m=1}^{3} A_m e^{i(\boldsymbol{k_m} \cdot \boldsymbol{x} - \omega_m t)} + c.c. \right] + \varepsilon^2 \left[\sum_{m=1}^{3} \sum_{n=-m}^{m} B_{mn}(t) e^{i((\boldsymbol{k_m} + \boldsymbol{k_n}) \cdot \boldsymbol{x} - (\omega_n + \omega_m) t)} + c.c. \right] + O(\varepsilon^3)$$

If we substitute this ansatz in the original set of equations $N(u) = 0$, we expect to recover the linearized equations to $O(\varepsilon)$. At $O(\varepsilon^2)$, we obtain a set of equations for the coefficients $B_{nm}(t)$, which, in a manner similar to the cases described in Lectures 5 and 6, have a LHS equal to the LHS of the homogeneous linearized equations, and a RHS which depends on the first order amplitudes A_m times a phase function $e^{i(\boldsymbol{k_m} \cdot \boldsymbol{x} - \omega_m t)}$. By construction of the resonant triad, some of these phases are in resonance with the LHS, leading to solutions $B_{mn}(t)$ which grow linearly with time. If that is the case, after a duration $t \sim 1/\varepsilon$, the $O(\varepsilon^2)$ terms are as important as the $O(\varepsilon)$ terms and the asymptotic expansion is no longer valid.

To solve the problem, we introduce as in Lectures 5 and 6 a slower timescale[1] in the wave amplitudes, perform a multiple-scale analysis, and find a compatibility condition to prevent the amplitudes from blowing up. Hence, let u now be:

$$u(\boldsymbol{x}, t; \varepsilon) = \varepsilon \left[\sum_{m=1}^{3} A_m(\varepsilon \boldsymbol{x}, \varepsilon t) e^{i(\boldsymbol{k_m} \cdot \boldsymbol{x} - \omega_m t)} + c.c. \right] + O(\varepsilon^2).$$

In the general case, as found by Benney & Newell (1967) ([2]) among others, the amplitudes have to satisfy this set of compatibility conditions in order to prevent any unphysical growth in the equations:

[1] and/or a longer lengthscale. . .

$$
\begin{aligned}
\partial_\tau A_1 + \boldsymbol{c_1} \cdot \nabla A_1 &= i\delta_1 A_2^* A_3^*, \\
\partial_\tau A_2 + \boldsymbol{c_2} \cdot \nabla A_2 &= i\delta_2 A_1^* A_3^*, \\
\partial_\tau A_3 + \boldsymbol{c_3} \cdot \nabla A_3 &= i\delta_3 A_1^* A_2^*,
\end{aligned} \tag{3}
$$

where $\tau = \varepsilon t$ is the slow timescale, $\boldsymbol{c_n} = \mathrm{d}\omega/\mathrm{d}k|_{k=k_n}$ is the group velocity of the n^{th} mode, and δ_n is a real-valued constant which depends on the nonlinear terms of the original set of equations $N(u) = 0$ and can be deduced from an analysis of the original Hamiltonian. These equations can be applied in various contexts, *e.g.* capillary-gravity waves ([13]) or χ_2 materials in optics ([3]). Equations (3) are the so-called PDE version of the equations of triad resonances.

3 Mathematical structure of a single triad of ODEs

A special case occurs when the complex wave amplitude depends on only one independent variable. This can happen for example for systems in which the spatial planform is fixed but its amplitude varies with time. In this case, equations (3) reduce to:

$$
\begin{aligned}
A_1' &= i\delta_1 A_2^* A_3^*, \\
A_2' &= i\delta_2 A_1^* A_3^*, \\
A_3' &= i\delta_3 A_1^* A_2^*,
\end{aligned} \tag{4}
$$

where A_n' is either $\partial_\tau A_n$ or $c_n \partial_x A_n$. These equations are the so-called ODE version of the equations of triad resonances.

3.1 Hamiltonian structure and integrability

If the original system is hamiltonian, so is this reduced system. The conjugate variables for the single-triad ODEs are

$$
\begin{aligned}
q_n(\tau) &= \mathrm{sign}(\delta_n) \frac{A_n(\tau)}{\sqrt{|\delta_n|}}, \\
p_n(\tau) &= \frac{A_n^*(\tau)}{\sqrt{|\delta_n|}} \text{ for } n = 1, 2, 3
\end{aligned}
$$

and the Hamiltonian of the system is

$$
\begin{aligned}
H &= i(A_1 A_2 A_3 + A_1^* A_2^* A_3^*) \\
&= i\sqrt{|\delta_1 \delta_2 \delta_3|}(\mathrm{sign}(\delta_1 \delta_2 \delta_3) q_1 q_2 q_3 + p_1 p_2 p_3).
\end{aligned}
$$

It is easy to verify that H and all the $\{p_n, q_n\}$ verify

$$
q_n' = \frac{\partial H}{\partial p_n} \quad \text{and} \quad p_n' = -\frac{\partial H}{\partial q_n} \quad \text{for } n = 1, 2, 3.
$$

The following quantities are constants of the motion:

$$
\begin{aligned}
-iH &= A_1 A_2 A_3 + A_1^* A_2^* A_3^*, & (5)\\
J_1 &= \frac{A_1 A_1^*}{\delta_1} - \frac{A_3 A_3^*}{\delta_3}, & (6)\\
J_2 &= \frac{A_2 A_2^*}{\delta_2} - \frac{A_3 A_3^*}{\delta_3}. & (7)
\end{aligned}
$$

(and so is the equivalently defined J_3, although J_3 is not independent of J_1 and J_2). Equations (6) and (7) are known as the Manley-Rowe equations.

We can now use some of the properties of Hamiltonian systems discussed in Lecture 4 to study the integrability of the problem. Recall that the notion of integrability first requires the definition of the Poisson bracket $\{F, G\}$ for any two real-valued functions $F(p_n, q_n, t)$ and $G(p_n, q_n, t)$:

$$
\begin{aligned}
\{F, G\} &= \sum_{m=1}^{3} \left(\frac{\partial F}{\partial p_m} \frac{\partial G}{\partial q_m} - \frac{\partial F}{\partial q_m} \frac{\partial G}{\partial p_m} \right) \\
&= \sum_{m=1}^{3} \delta_m \left(\frac{\partial F}{\partial A_m^*} \frac{\partial G}{\partial A_m} - \frac{\partial F}{\partial A_m} \frac{\partial G}{\partial A_m^*} \right).
\end{aligned}
$$

We have already identified 3 constants of motion for our 6-dimensional phase space. We next have to verify that these constants of motion are *in involution*, i.e. that any two pairs have a null Poisson Bracket. This can indeed be verified, as for example:

$$
\begin{aligned}
\{-iH, J_1\} &= \sum_{m=1}^{3} \delta_m \left(\frac{\partial(-iH)}{\partial A_m^*} \frac{\partial J_1}{\partial A_m} - \frac{\partial(-iH)}{\partial A_m} \frac{\partial J_1}{\partial A_m^*} \right) \\
&= (A_1^* A_2^* A_3^* - A_1 A_2 A_3) + 0 + (-A_1^* A_2^* A_3^* + A_1 A_2 A_3) \\
&= 0.
\end{aligned}
$$

It can be easily shown in a similar way that $\{-iH, J_2\} = \{J_1, J_2\} = 0$. Moreover, any 3 independent linear combination of these constants of motion will then also be *in involution*. All that remains to be done is to construct the 3 action variables P_n ($n = 1, 2, 3$) as linear combinations of $-iH$, J_1 and J_2, together with the 3 conjugate angle variables, such that the $\{P_n, Q_n\}$ pairs satisfy

$$
\begin{aligned}
\frac{\partial H}{\partial Q_n} &= -\frac{\mathrm{d}P_n}{\mathrm{d}t} = 0 \\
\frac{\partial H}{\partial P_n} &= \frac{\mathrm{d}Q_n}{\mathrm{d}t}
\end{aligned}
$$

The resultant system is completely integrable. As we saw in Lecture 4, the 3 action variables define a 3-D surface in the 6-D phase space. Every solution of the ODEs consists of straight-line motion on this surface.

The topology of the 3-D surface depends uniquely on the signs of δ_1, δ_2 and δ_3. For the Hamiltonian system defined above, it can be shown that the general solution can be written in terms of elliptic functions. In this case, the surface defined by $\{-iH, J_1, J_2\}$ is compact if and only if $\{\delta_1, \delta_2, \delta_3\}$ do not all have the same sign. To see this, first note that if the signs of δ_1 and δ_3 are different for example, then $J_1 =$ constant is the equation for an ellipse in the $(|A_1|, |A_3|)$ plane. It follows that provided at least one of the δ_n has a different sign, it is always possible to select a pair among the three possible J_n for which the 2 equations J_i=const and J_j=const combined describe a torus.

In the unusual situation however where δ_1, δ_2 and δ_3 all have the same sign, Coppi, Rosenbluth & Sudan (1969) [7] showed that A_1, A_2 and A_3 can all blow up together, in finite time: indeed, as J_1 is a constant of the motion, if $|A_1|$ increases, $|A_3|$ has to increase also according to equation (6) and as J_2 is also a constant of the motion, $|A_2|$ has to increase as well... This is the explosive instability, which is discussed in Lecture 21. It has applications in plasma physics ([9]), density-stratified shear flows ([6, 8]) and for vorticity waves ([14]).

3.2 Properties of single-triad systems

Consider a single triad of ODEs, without dissipation. In this example we consider a system which has coefficients such that $\delta_1 > 0$, $\delta_2 > 0$ and $\delta_3 < 0$.

One property of such a configuration is that only A_3 can share energy with the other modes. This can be readily seen from the definition of J_1 and J_2 in equations (6) and (7), which can be rewritten here as:

$$J_1 = \frac{|A_1|^2}{|\delta_1|} + \frac{|A_3|^2}{|\delta_3|} \quad \text{and} \quad J_2 = \frac{|A_2|^2}{|\delta_2|} + \frac{|A_3|^2}{|\delta_3|}.$$

Therefore, if $|A_1|$ (respectively $|A_2|$) initially has almost all the energy, then J_2 (respectively J_1) is initially small. Since J_2 (J_1) is a constant of motion, it necessarily remains small thereby limiting the amplitude of the other modes. By contrast, if $|A_3|$ starts with almost all the energy, it can distribute it to the whole system.

Moreover, Hasselmann (1967) [10] proved that in a single triad of ODEs in which $\{\delta_1, \delta_2, \delta_3\}$ do not all have the same sign, the interaction coefficient δ_n with the opposite sign from the other two (in our example, that would be $|A_3|$) is always the wave mode with the highest frequency in the triad.

This section has shown that the case of a single triad of ODEs is well understood. If the complex wave amplitude do *not* depend on only one independent variable, we have to move on to the case of a single triad of PDEs.

4 Triads of PDEs

Zakharov & Manakov (1976) [17] showed that the system of equations (3) of PDEs is completely integrable. Kaup (1978) [11] partly solved the initial-value problem in 1-D on $-\infty < x < \infty$, with restrictions. Kaup, Reiman & Bers (1980) solved the initial-value problem in 3-D in all space, under some restrictions. Besides that, few physical applications of this theory have been developed so far.

Zakharov (1968) [16] went further than a single resonant triad interaction of PDEs and considered all possible interactions, including the non-resonant ones. This led to the Zakharov's integral equation for the amplitudes:

$$\partial_t A(\boldsymbol{k}) + i\omega(\boldsymbol{k})A(\boldsymbol{k}) = -i\iint \left[V(\boldsymbol{k},\boldsymbol{k_1},\boldsymbol{k_2})\delta(\boldsymbol{k}+\boldsymbol{k_1}+\boldsymbol{k_2})A^*(\boldsymbol{k_1})A^*(\boldsymbol{k_2}) + perm.\right] d\boldsymbol{k_1}d\boldsymbol{k_2}$$
$$-i\iiint \left[W(\boldsymbol{k},\boldsymbol{k_1},\boldsymbol{k_2},\boldsymbol{k_3})\delta(\boldsymbol{k}+\boldsymbol{k_1}+\boldsymbol{k_2}+\boldsymbol{k_3})A^*(\boldsymbol{k_1})A^*(\boldsymbol{k_2})A^*(\boldsymbol{k_3}) + perm.\right] d\boldsymbol{k_1}d\boldsymbol{k_2}d\boldsymbol{k_3}. \tag{8}$$

V and W are generalized interaction coefficients and $perm.$ stands for the permutations in the combinations of $\boldsymbol{k}$'s, *e.g.* $\delta(\boldsymbol{k} \pm \boldsymbol{k_1} \pm \boldsymbol{k_2})\ldots$ This equation acts on the fast timescale whereas equations (3) acted on the slow timescale τ. As the slow timescales are still present in equation (8), everything has to be resolved and a numerical integration is therefore slow and expensive.

In general, single triads of ODEs are insufficient (i) when wave envelopes have spatial variability, in which case PDEs are required, (ii) when there are multiple triad interactions, in which case more ODEs (or PDEs) are required and (iii) when dissipation occurs, in which case one needs to study non-Hamiltonian ODEs (the case of a single dissipative triad is studied in Lecture 20). Having discussed a few properties of single triads of PDEs, let us now discuss problems associated with multiple triad interactions under experimental conditions.

5 Application to capillary-gravity waves

In Section 2.1, we saw that for capillary-gravity waves (and most realistic systems) there exists a continuum of triads. So one can rightfully wonder whether any of the results described in Section 3 for single-triad resonance remain applicable. If not, under what conditions is it possible to model the problem with just a small number of triads? Under what conditions are we instead forced to consider all possible triads, as in Zakharov's integral formulation seen in Section 4? This section discusses these questions in the light of experimental results.

Simmons (1969) [15] conjectured that the magnitudes of the interaction coefficients δ_n do not vary much across the continuum of possible capillary-gravity waves resonant triads. Should this be correct, then any energy input into a single wave mode will eventually be transferred to all the other modes, triad by triad, thus generating a broad-banded response of the system to the applied forcing.

Perlin, Henderson and Hammack(1990) [12] attempted to test this conjecture experimentally. In their work, a tank of typical size $10\,\mathrm{cm} \times 1\,\mathrm{m}$ filled with water is forced with a paddle oscillating at a frequency $f_0 = 25\,\mathrm{Hz}$, therefore exciting capillary-gravity waves that are subject to triad interactions. As we have seen in Section 2.1, in 2D and unbounded configurations, the spectrum of possible interactions is a continuum. Although this geometry is bounded, it is large enough to have a significant number of possible modes and one can therefore expect to have a broad response in the frequency space.

A first series of experiments was performed and contrary to expectations did not display the expected broad-band response, as can be seen in Figure 2. All figures show that, no matter what the forcing amplitude is, the response of the fluid is localized in a few modes with frequency mainly around f_0. Figures 2(a) and 2(b) show results of the experiment with two different forcing amplitudes. The components that are to be seen in this picture are:

- a strong signal at f_0,
- a second harmonic at $f_0 + f_0 = 50\,\text{Hz}$,
- a third harmonic at $2f_0 + f_0 = 75\,\text{Hz}$,
- a very weak component at $f_c = 60\,\text{Hz}$ that can only be seen in Figure 2(b). It is an internal frequency of the computer that controls the forcing mechanism and that is transmitted to the tank as mechanical vibration.
- a frequency $f_\alpha = 35\,\text{Hz} = f_0 - f_c$,
- two frequencies $f_\beta = 10\,\text{Hz}$ and $f_\gamma = 15\,\text{Hz}$, characterized by $f_\beta + f_\gamma = f_0$.

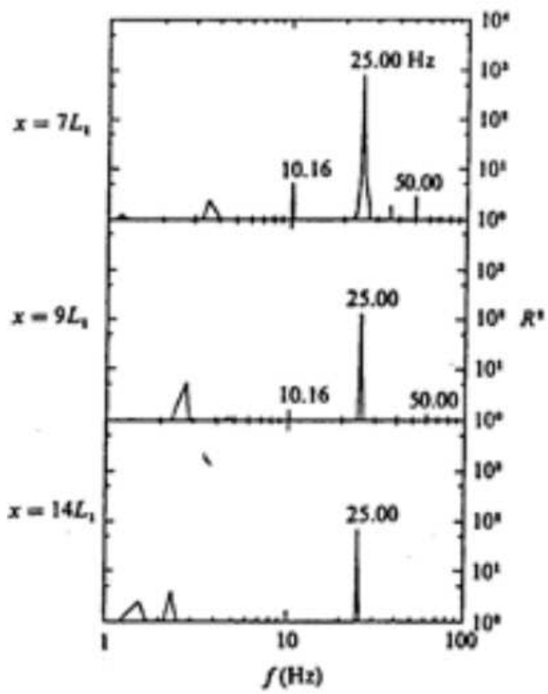
If we ignore the second and third harmonics of the forcing, the specific transmission chain is then the following: f_c and f_0, the forcing frequencies, interact to form f_α, then f_0 and f_α interact to form f_β which then interacts with f_0 to form f_γ.

A second series of experiments was later performed with a newer computer which did not perturb the system with the additional frequency at $f_c = 60\,\text{Hz}$. In these new experiments, a broad frequency spectrum was observed in response to the forcing. It is therefore quite remarkable to note how the whole system dynamics change depending on the presence or absence of the additional forcing at f_c, even when the forcing is so weak as to be barely detected in Figure 2. The difference between the two sets of experiments suggests the presence of a selection mechanism for individual triads through the additional forcing.

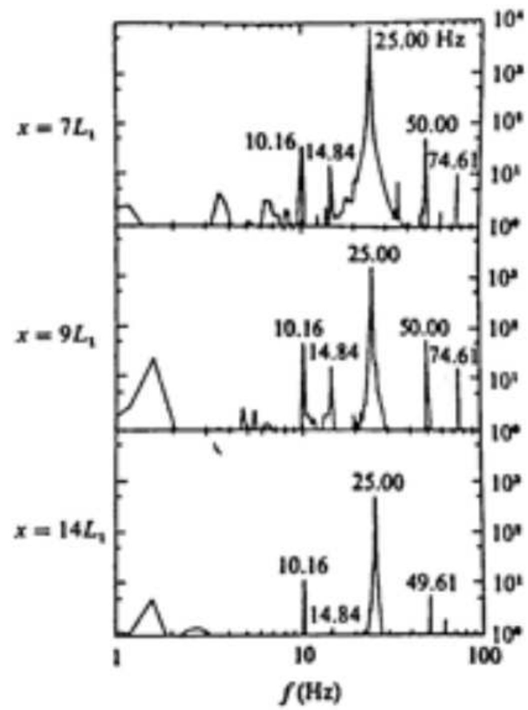
Now let us focus on the experiment related to Figure 2. In the triad involving $f_\alpha = f_c - f_0$ and following the notations of Section 3.2, A_1, A_2 and A_3 correspond to f_0, f_α and f_c respectively. A_3 is the highest frequency but A_1 has almost all the energy initially, so A_2 and A_3 should remain small. Although they indeed remain small compared to A_1, A_2 gets significantly bigger than A_3, in contradiction with Section 3.2. In the triad involving $f_\alpha - f_0 = f_\beta$, (with A_1, A_2 and A_3 corresponding to f_0, f_β and f_α this time), again we have $A_3 \ll A_1$, but A_2 gains some energy in the process. Only in the last triad involving $f_0 - f_\beta = f_\gamma$ is the highest mode also the most energetic one. Hasselmann's results presented in Section 3.2, which were valid for a single triad, are clearly not relevant here.

These experiments have revealed two surprising features: the selection of a small number of triads among the possible continuum by a weak periodic signal, and the invalidation of single-triad theory (see Section 3.2) as soon as more than one triad is involved. In an actual physical problem involving 3-wave interactions, these two conclusions show how difficult it may be, in practice, to make even qualitative predictions about a system's response to forcing. In general, it is indeed very hard to predict whether a single, a few, or all triads must be taken into account. Any process has to be investigated in detail before answering this question, following Prof. Einstein[2]:

[2] **Albert Einstein** (14 March 1879, Ulm, Germany – 18 April 1955, Princeton, USA) German-born scientist who was somehow active as a theoretical physicist but who unfortunately died four years too soon to benefit from any Geophysical Fluid Dynamics summer course. If it had been the case, which could have been possible given the quality of his resume, there is no doubt that he could have achieved much more.



(a) Lower amplitude



(b) Higher amplitude

Figure 2: Periodograms resulting from the excitation of capillary-gravity waves. The forcing frequency is $f_0 = 25\,\mathrm{Hz}$. The figures from top to bottom show the field at various locations, further away from the forcing apparatus.

A good mathematical model of a physical problem should be as simple as possible, and no simpler.

6 Summary and conclusion

Among all the nonlinear phenomena studied in this summer course, triad interactions are, at first glance, the simplest, most natural and most intuitive ones. This lecture introduced the mathematical theory of triad interactions, beginning with the simplest possible example, namely the single triad of ODEs. It is interesting to note that despite its "simplicity", the problem of single triads of ODEs was only solved approximately at the same time as the first results on solitons were published. The cases of a single triad of PDEs, and/or multiple triad interactions, were solved later and are very briefly described here.

In the last part of these notes, theoretical results are confronted to an experimental situation which sheds light on the difficulty in isolating single-triad interactions in practice. These results should cast some doubt on the belief that these problems are simple, if such a belief ever existed.

Triad interactions have a lot in common with other topics discussed in this lecture course: for example, the KdV model seen in Lecture 5 could be seen as a triad interaction between spatial frequencies 0, 0 and 0: $0 + 0 = 0$. Another more explicit example is the NLS model (see Lectures 3 and 13), which can also be seen as the interaction between a wave of spatial frequency k and a wave of spatial frequency $k + \delta k$ close to k.

This problem, even in its most simple configurations such as single triads or simple triad clusters, is still the center of active mathematical attention (see *e.g.* Bustamante & Kartashova (2009) [4, 5]).

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