1 2-D Convection-Diffusion

Consider a 2-D divergence-free, periodic, steady flow field \( u(t, x) \) in a domain without any boundaries. Let \( \tilde{\rho}(t, x) \) be the concentration of a passive scalar, say temperature. Then the non-dimensional governing equations for the non-dimensional variables \( \tilde{\rho} \) and \( u \) are:

\[
\begin{align*}
\tilde{\rho}_t + u \cdot \nabla \tilde{\rho} &= \epsilon \Delta \tilde{\rho}, \\
\nabla \cdot u &= 0,
\end{align*}
\]

(1)

(2)
together with the initial condition \( \tilde{\rho}(0, x) = \tilde{\rho}^0(x) \). Note that \( \epsilon \) is dimensionless parameter since \( \epsilon^{-1} \sim UL/\nu = Pe \), where \( Pe \) is the Peclet number and \( L \) is the size of the periodic cell. By integrating (1) over \( \mathbb{R}^2 \) and using (2), we see that if \( \int_{\mathbb{R}^2} \tilde{\rho}^0(x) dx = 1 \), then \( \int_{\mathbb{R}^2} \tilde{\rho}(t, x) dx = 1 \). Also if \( \tilde{\rho}^0(x) \geq 0 \), then \( \tilde{\rho}(t, x) \geq 0 \). Since \( \nabla \cdot u = 0 \) and the flow is 2-D, it is possible to introduce a stream function \( \psi(x) \):

\[
u = (-\psi_y, \psi_x).
\]

(3)

If \( \psi(x, y) = \sin x \sin y + \delta \cos x \cos y \), then we have a cellular flow if \( \delta = 0 \), and a shear flow if \( \delta = 1 \). Since \( x(t) \) is the position of a diffusing particle, the evolution equation for \( x(t) \) can be written as the following SDE:

\[
dx(t) = u(x(t)) dt + \sqrt{2\epsilon} \, dW(t).
\]

(4)

If there is no diffusion (i.e. there is no \( \sqrt{2\epsilon} \, dW(t) \) term in (4)), a particle starting on a particular streamline remains on the streamline. If we have diffusion, there is a possibility for a particle which starts in the region \((a)\) to move to the region \((b)\) (See Figure 1). In that case, \( \tilde{\rho} \) can be interpreted as the probability density of \( x(t) \).
2 Effective diffusivities

Consider a diffusing particle, \( \lim_{t \to \infty} \frac{1}{2t} \mathbb{E} \left[ \{(x(t) - x(0)) \cdot e\}^2 \right] \) always exists for \( u(x) \) which is periodic and satisfies \( \nabla \cdot u = 0 \) and \( \langle u(x) \rangle = 0 \), where \( \langle \cdot \rangle \) represents the periodic cell average. We denote this limit as \( \sigma^*_e(e) \), so called the effective diffusivity. It is a quadratic form of \( e \).

We now take the large time, long distance limit of the PDE (1) by changing the variables \( t \to n^2 t, x \to nx \) and letting \( n \to \infty \). (This process is called the homogenization.) \( \rho_n(t, x) = \rho(n^2 t, nx) \) converges to \( \rho(t, x) \) in an appropriate sense as \( n \to \infty \), where \( \rho(t, x) \) is the solution of the homogenized equation

\[
\rho_t = \nabla \cdot (\sigma^*_e \nabla \rho),
\]

with \( \rho(0, x) = \rho^0(x) \). \( \sigma^*_e(e) \) is a constant matrix, or more precisely

\[
\sigma^*_e(e) = \langle (\epsilon I + \Psi)(\nabla \chi + e) \cdot e \rangle,
\]

where \( I \) is the identity matrix, \( \chi(x) \) is a periodic function in \( \mathbb{R}^2 \), and

\[
\Psi(x) = \begin{pmatrix}
0 & -\psi(x, y) \\
\psi(x, y) & 0
\end{pmatrix}.
\]

It is found that \( \sigma^*_e(e) \) satisfies the polarization relation

\[
(\sigma^*_e)_{ij} = \frac{1}{4} [\sigma^*_e(e_i + e_j) - \sigma^*_e(e_i - e_j)], \quad i, j = 1, 2,
\]

where \( e_1 = (1, 0) \), \( e_2 = (0, 1) \). Apart from the homogenized equation (5), the homogenization process also yields the cell problem, that is

\[
\nabla \cdot [(\epsilon I + \Psi(x)(\nabla \chi + e)] = 0.
\]
\( \sigma^* \) can be calculated by solving (9) for \( \chi \) and plugging it into (6). The full derivation of (5), (6) and (9) will be shown in the next section. The physical interpretation of \( \sigma^*(e) \) is the average flux in the direction \( e \) when there is a unit average gradient in the direction \( e \).

### 3 Asymptotics for \( \rho_n(t, x) \)

Recall the passive scalar advection equation in the fast variables

\[
\frac{\partial \rho_n}{\partial t} = \nabla \cdot ([I + \Psi_n(x)] \nabla \rho_n) \tag{10}
\]

with initial condition

\[
\rho_n(0, x) = \rho^0(x)
\]

where \( I \) is the identity matrix, \( \Psi_n \) was defined previously, and we have set \( \epsilon = 1 \). First we must check that (10) solves (1)

\[
\frac{\partial \rho_n}{\partial t} = \left( \frac{\partial \rho_n}{\partial x} + \psi_n \frac{\partial \rho_n}{\partial y} \right)_x + \left( \frac{\partial \rho_n}{\partial y} - \psi_n \frac{\partial \rho_n}{\partial x} \right)_y = (\rho_n)_{xx} + (\rho_n)_{yy} - (\psi_n)_x (\psi_n)_y + (\psi_n)_y (\psi_n)_x - \psi_n (\rho_n)_{xy} + \psi_n (\rho_n)_{yx} = \Delta \rho_n - u \cdot \nabla \rho_n.
\]

Next we expand \( \rho_n \) in an asymptotic series

\[
\rho_n(t, x) = \rho(t, x) + \frac{1}{n} \rho^{(1)}(t, x, nx) + \frac{1}{n^2} \rho^{(2)}(t, x, nx) + ...
\]

It is clear that for this problem we have a clean separation of scales. The fast time scale does not appear because the coefficients are time homogeneous.

Let \( nx = \xi \) so that \( \nabla \rightarrow \nabla_x + n \nabla_\xi \). Plugging \( \rho_n \) into (10) we get

\[
\frac{\partial}{\partial t} \left( \rho + \frac{1}{n} \rho^{(1)} + \frac{1}{n^2} \rho^{(2)} + ... \right) = \nabla_\xi \cdot \left[ (I + \Psi_n(\xi)) \cdot (\nabla_x + n \nabla_\xi) \left( \rho + \frac{1}{n} \rho^{(1)} + \frac{1}{n^2} \rho^{(2)} + ... \right) \right].
\]

As is standard procedure, we equate the coefficients for powers of \( n \). At \( O(n^2) \):

\[
\nabla_\xi \cdot [(I + \Psi_n(\xi)) \nabla_\xi \rho] = 0. \tag{11}
\]

Note (11) is automatically satisfied since \( \rho \) is not a function of \( \xi \). At \( O(n) \):

\[
\nabla_\xi \cdot [(I + \Psi_n(\xi)) \nabla_\rho] + \nabla_x \cdot [(I + \Psi_n(\xi)) \nabla_\xi \rho] + \nabla_\xi \cdot \left[ (I + \Psi_n(\xi)) \nabla_\xi \rho^{(1)} \right] = 0. \tag{12}
\]
The second term in (12) is zero via (11). Upon rewriting (12) we get
\[
\nabla_\xi \cdot \left[ (I + \Psi_n(\xi)) \left( \nabla_\xi \rho^{(1)} + \nabla_x \rho \right) \right] = 0
\]
which resembles the cell problem (9). Equation (13) is a PDE for \(\rho^{(1)}(\xi)\) (periodic in \(\xi\)). We can cast (13) into the cell problem by letting
\[
\rho^{(1)}(t, x, \xi) = \sum_{j=1}^{d} \chi_{e_j}(\xi) \frac{\partial \rho}{\partial x_j}(t, x)
\]
which separates the \(\xi\) dependence from the \(t, x\) dependence. The function \(\chi_{e}(\xi)\) satisfies
\[
\nabla_\xi \cdot \left[ (I + \Psi_n(\xi)) \left( \nabla_\xi \chi_{e}(\xi) + e \right) \right] = 0.
\]
At \(O(1)\):
\[
\frac{\partial \rho}{\partial t} = \nabla_\xi \cdot \left[ (I + \Psi_n(\xi)) \nabla_x \rho^{(2)} \right] + \nabla_\xi \cdot \left[ (I + \Psi_n(\xi)) \nabla_x \rho^{(1)} \right] + \nabla_x \cdot \left[ (I + \Psi_n(\xi)) \nabla_\xi \rho^{(1)} \right] + \nabla_x \cdot \left[ (I + \Psi_n(\xi)) \nabla_x \rho \right] - \frac{\partial \rho}{\partial t}.
\]
which is a PDE for \(\rho^{(2)}(\xi)\) (periodic in \(\xi\)) with \(t, x\) as parameters. This can be re-written as
\[
\nabla_\xi \cdot \left[ (I + \Psi_n(\xi)) \nabla_\xi \rho^{(2)} \right] + S = 0
\]
where
\[
S = \nabla_\xi \cdot \left[ (I + \Psi_n(\xi)) \nabla_x \rho^{(1)} \right] + \nabla_x \cdot \left[ (I + \Psi_n(\xi)) \nabla_\xi \rho^{(1)} \right] + \nabla_x \cdot \left[ (I + \Psi_n(\xi)) \nabla_x \rho \right] - \frac{\partial \rho}{\partial t}.
\]
Upon taking the cell average of (15), we obtain
\[
\left\langle \nabla_x \cdot \left[ (I + \Psi_n(\xi)) \nabla_\xi \rho^{(2)} \right] \right\rangle + \langle S \rangle = 0
\]
and since \(\nabla_\xi \rho^{(2)}\) is a gradient of a periodic function, \(\langle S \rangle = 0\) which yields
\[
\frac{\partial \rho}{\partial t} = \left\langle \nabla_\xi \cdot \left[ (I + \Psi_n(\xi)) \nabla_x \rho^{(1)} \right] \right\rangle + \left\langle \nabla_x \cdot \left[ (I + \Psi_n(\xi)) \nabla_\xi \rho^{(1)} \right] \right\rangle + \langle S \rangle = \nabla_x \cdot \left\langle \left[ (I + \Psi_n(\xi)) \left( \nabla_\xi \rho^{(1)} + \nabla_x \rho \right) \right] \right\rangle.
since $\nabla_x \rho^{(1)}$ is the gradient of a periodic function. In component form

$$\frac{\partial \rho}{\partial t} = \sum_{i,j} \frac{\partial }{\partial x_i} \left\langle A_{ij}(\xi) \left( \frac{\partial \rho^{(1)}}{\partial \xi_j} + \frac{\partial \rho}{\partial x_j} \right) \right\rangle$$

$$= \sum_{i,j} \frac{\partial}{\partial x_i} \left\langle A_{ij}(\xi) \left( \sum_k \chi_{e_k}(\xi) \frac{\partial \rho(t,x)}{\partial x_k} + \frac{\partial \rho}{\partial x_j} \right) \right\rangle$$

$$= \sum_{i,j} \sum_k \left\langle A_{ij} \left( \frac{\partial \chi_{e_k}}{\partial \xi_j} \frac{\partial^2 \rho}{\partial x_i \partial x_k} + \delta_{jk} \frac{\partial^2 \rho}{\partial x_i \partial x_k} \right) \right\rangle$$

$$= \sum_{i,k} \left( \sum_j \left\langle A_{ij} \left( \frac{\partial \chi_{e_k}}{\partial \xi_j} + \delta_{jk} \right) \right\rangle \right) \frac{\partial^2 \rho}{\partial x_i \partial x_k}$$

where $A_{ij} = I_{ij} + \Psi_{ij}(\xi)$. Thus, we obtain the homogenized equation

$$\frac{\partial \rho}{\partial t} = \sum_{i,k} \sigma^*_{ik} \frac{\partial^2 \rho}{\partial x_i \partial x_k}$$

(17)

with

$$\sigma^*_{ik} = \sum_j \left\langle A_{ij} \left( \frac{\partial \chi_{e_k}}{\partial \xi_j} + \delta_{jk} \right) \right\rangle$$

or

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (\sigma^* \nabla \rho).$$

(18)

In summary, the key ideas for homogenization are:

1) Perform a multiscale expansion

$$t, x \sim \text{macroscopic scales} \quad \text{slow}$$

$$n^2 t, n x \sim \text{microscopic scales} \quad \text{fast}$$

the resulting PDE will involve both fast and slow variables. In our case $\psi \to \psi(n x)$. In general $\psi \to \psi(n^2 t, n x, t, x)$.

2) Seek an expansion in which the principle term is slowly varying $(t, x)$.

3) The coefficients of the slowly varying equation come from a cell problem. In this case the term of interest was $\rho$ and we had to go to $O(1)$ to get the cell problem.
The effective diffusivity matrix, $\sigma^*_{\epsilon}$, is given by

$$
\sigma^*_{\epsilon} = \langle (\epsilon I + \Psi) (\nabla \chi) \cdot e \rangle \\
= \langle (\epsilon I + \Psi) (\nabla \chi + e) \cdot (\nabla \chi + e) \rangle
$$

where we added $\nabla \chi$ to $e$ because $\nabla \cdot [(\epsilon I + \Psi) (\nabla \chi + e)] = 0$ (9). Also since $(\nabla \chi + e) \cdot (\nabla \chi + e)$ is a quadratic form and $\Psi$ is skew symmetric, we obtain

$$
\sigma^*_{\epsilon}(e) = \epsilon \langle |\nabla \chi + e|^2 \rangle \\
= \epsilon + \epsilon \langle |\nabla \chi|^2 \rangle.
$$

From this it is clear that convection always enhances diffusion since $\sigma^*_{\epsilon}(e) \geq \epsilon$.

Finally we check convergence of the asymptotic expansion

1) 

$$
\max_{0 \leq t \leq T, x \in \mathbb{R}^2} |\rho_n(t,x) - \rho(x,t)| \leq \max_{0 \leq t \leq T, x \in \mathbb{R}^2} \left| \frac{1}{n} \rho^{(1)} + O\left(\frac{1}{n}\right) \right| \leq C_T \frac{1}{n}
$$

provided $\rho_0$ decays rapidly at infinity and is smooth.

2) 

$$
\int_0^\infty \int_{\mathbb{R}^2} (\nabla \rho_n - \nabla \rho) \theta(t,x) dx dt \to 0
$$

where $\theta$ is a test function. This says that on average the gradient converges. Calculating $\nabla \rho_n$ we obtain

$$
\nabla \rho_n = \nabla \rho + \nabla \xi \rho^{(1)}(t,x,nx) + ...
$$

3) 

$$
\sup_{0 \leq t \leq T} \int_{\mathbb{R}^2} |\nabla \rho_n - (\nabla \rho + \nabla \xi \rho^{(1)})|^2 dx \leq C_T \frac{1}{n}
$$

thus $\rho^{(1)}$ closes the problem and allows us to determine $\nabla \rho_n$. Note that 3) implies 2).

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