Lecture 7: Tidal Bores

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1 Introduction

In this lecture we discuss the formation of tidal bores in rivers. We discuss the effect of nonlinear terms in the shallow water equations. The convergence speed is obtained for tidal waves. This can be used to estimate the run length before a bore is formed. In §2 the nonlinear shallow water equations are discussed. We discuss a method of solution of the nonlinear shallow water equations in §3. A convergence speed is obtained in §4 and finally in §5 we discuss the Korteweg-de Vries Equation.

2 Nonlinear shallow water equations

The shallow water equations with nonlinear terms can be written as,

$$u_t + uu_x + g\zeta_x = 0, \tag{1}$$

$$\zeta_t + \frac{\partial[(h+\zeta)u]}{\partial x} = 0. \tag{2}$$

The effect of nonlinear term can be better understood using the following model equation,

$$u_t + (u+c)u_x = 0, \tag{3}$$

where c is constant. To obtain solution of (3) we can write, $u = \epsilon u_1 + \epsilon^2 u_2$. In the lowest order of ϵ we just obtain the simple wave equation,

$$u_{1t} + cu_{1x} = 0. \tag{4}$$

The solution of (4) is given as,

$$u_1 = F(x-ct). \tag{5}$$

If we consider equation at $O(\epsilon^2)$ then we get,

$$u_{2t} + cu_{2x} = -u_1 u_{1x}. \tag{6}$$

Consider an initial wave of sinusoidal form such that $u_1 = a\sin(k(x-ct))$. By substitution of u_1 in (6) we get,

$$u_{2t} + c\ u_{2x} = -\frac{1}{2}a^2 k \sin(2k(x-ct)), \tag{7}$$

with initial condition $u_2 = 0$ at $t = 0$. By solving (7) we get,

$$u_2 = -\frac{1}{2} k a^2 t \sin(2k(x - ct)). \tag{8}$$

When we are solving for u_2 in (7), this is a system of natural frequency c and is forced with a forcing frequency c. The value of u_2 is such that it increases linearly with time. This is counter intuitive because u_2 is $O(\epsilon^2)$ term and u_1 is $O(\epsilon)$ term, so u_1 should dominate over u_2, but this clearly cannot persist for long time.

An alternate interpretation of (3) can be in terms of its characteristics. The characteristic curves of (3) are given by the solutions of the ODEs,

$$\frac{dx}{dt} = u + c, \tag{9}$$

$$\frac{du}{dt} = 0. \tag{10}$$

That is, u is constant along each characteristic, which is the straight line,

$$x = x_0 + (u + c)t, \tag{11}$$

where x_0 is a constant labeling the various curves (by initial position). At $t = 0, u = a \sin(kx_0)$. Since this is a property of each characteristic, the solution is given implicitly by,

$$u = a \sin[k(x - ut - ct)]. \tag{12}$$

Solution (12) can be verified by substitution in (3). Evidently, the larger the amplitude, the faster the disturbance propagates with the result that the crests of the sinusoid travel faster than the troughs, which inexorably steepens the forward facing slopes. Eventually the crests catch the troughs and (12) predicts a multi-valued solution, which is unphysical. Instead, one must impose additional physics (such as dissipation) to regularize the problem, the outcome of which is typically the prediction of the formation of a shock.

3 Characteristics of the nonlinear shallow water equations

The nonlinear shallow water equations (1)-(2) can be rewritten taking $c^2 = g(h + \zeta)$ so that

$$u_t + u u_x + \frac{\partial c^2}{\partial x} = 0, \tag{13}$$

$$\frac{\partial c^2}{\partial t} + \frac{\partial (c^2 u)}{\partial x} = 0. \tag{14}$$

By adding and subtracting (13) and (14) we get,

$$\frac{\partial (u + 2c)}{\partial t} + (u + c)\frac{\partial (u + 2c)}{\partial x} = 0, \tag{15}$$

$$\frac{\partial (u - 2c)}{\partial t} + (u - c)\frac{\partial (u - 2c)}{\partial x} = 0. \tag{16}$$

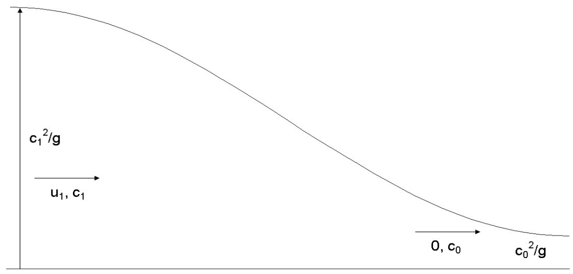

Figure 1: Steepening of tides causes formation of bores upstream in rivers. The distance upstream where bore forms can be calculated from the convergence speed.

The quantities $J_+ = u + 2c$ and $J_- = u - 2c$ are "Riemann invariants", each of which is constant along a particular characteristic curve on the (x,t) plane. J_+ is constant along the curves C_+ defined by,

$$\frac{dx}{dt} = u + c. \tag{17}$$

Similarly, J_- is constant along the curves C_- defined by,

$$\frac{dx}{dt} = u - c. \tag{18}$$

We can solve (13) and (14) by moving the Riemann invariants along the curves defined by C_+ and C_-. For most initial conditions, this procedure furnishes a simple numerical scheme. However, there are also special examples (such as the classic dam-breaking problem) that can be dealt with analytically.

4 Convergence Speed

We start by assuming that $J_- = B$ (constant everywhere) and J_+ is constant along the curves of C_+. For the rear and front of the wave we can write (see figure 1)

$$u_1 - 2c_1 = B, \tag{19}$$
$$-2c_0 = B. \tag{20}$$

From the equations above we can then write

$$u_1 = 2(c_1 - c_0). \tag{21}$$

The rear of the tidal wave travels at speed $u_1 + c_1$ and the front travels at speed c_0. So the convergence speed $u_{converg}$ is,

$$u_{converg} = 3(c_1 - c_0). \tag{22}$$

The steepening of the tidal wave occurs due to difference in velocity u and difference in heights c. Only one-third of the convergence speed is due to the difference in heights, the rest is due to the difference in velocity.

5 Korteweg-de Vries Equation

The Korteweg-de Vries equation can be found useful in a wide variety of wave problems in many diverse fields. In the ocean it can be found as a nonlinear representation of the shallow water equations. Thus, it can be used to model tidal waves. Its dimensional form is given as,

$$u_t + \left(\frac{3}{2}u + c_0\right)\frac{\partial u}{\partial x} + \frac{1}{6}c_0 h^2 \frac{\partial^3 u}{\partial x^3} = 0. \tag{23}$$

The 3/2 fraction in the non-linear term is to obtain a correct amount of steepening of the waves (based on the discussion in §4, one third of the steepening is due to convergence speed and the other two thirds is due to the difference in heights) and c_0 is the wave speed in shallow water. The last term in (23) is to make the equation consistent with the dispersion relation of the linear gravity waves given by,

$$\omega^2 = gk\tanh(kh) \approx c_0 k(1 - \frac{1}{6}k^2h^2). \tag{24}$$

The linearized shallow water approximation is useful only if the following two ratios are small:

$$\mu \equiv kh \ll 1, \qquad \epsilon \equiv \frac{A}{h} \ll 1. \tag{25}$$

Because of the severity of the the second restriction nonlinear theory of shallow water waves is needed. In order to derive this equation we first need to derive a low order Boussinesq-type equation. Since the velocity potential is analytic, we may expand it as a power series in the vertical coordinate. By substituting this series into the Laplace equation and by using the impermeable kinematic horizontal bottom boundary condition we can derive low order nonlinear shallow water equations [1].

$$\phi_{\mathrm{tt}} - \phi_{\mathrm{xx}} = \frac{\mu^2}{3}\phi_{\mathrm{xxtt}} - \epsilon\left(\phi_x^2 = +\frac{1}{2}\phi_t^2\right)_t. \tag{26}$$

Now let us assume a propagating wave in the positive x direction with a nondimensional wave speed of one and with a slow time variation: $\sigma = x - t, \tau = \epsilon t$. In terms of these variables, the derivatives become

$$\tfrac{\partial}{\partial x} \to \tfrac{\partial}{\partial \sigma}, \quad \tfrac{\partial}{\partial \sigma} \to -\tfrac{\partial}{\partial \sigma} + \epsilon\tfrac{\partial}{\partial \tau}. \tag{27}$$

By substituting these into the low order Boussinesq equation we get a nondimensional KDV equation in non-stationary coordinates

$$u_\tau + \frac{3}{2}uu_\sigma + \frac{\mu}{6\epsilon}u_{\sigma\sigma\sigma} = O(\epsilon, \mu^2). \tag{28}$$

Notes by Vineet K. Birman and Yaron Toledo.

References

[1] C. Mei, *The Applied Dynamics of the Ocean Surface Waves* (Teaneck, N.J. : World Scientific, Singapore, 1989).