GFD 2006 Lecture 7: Convection in mushy layers

Grae Worster; notes by Ian Eisenman and Victor Tsai

March 15, 2007

1 Convection Setup

Here we perform the analysis governing the regions 5 and 6 of the table shown in Lecture 6. Under these conditions we expect convection to occur within the mushy layer. We note that under strong convection, channels form within the mushy layer and are called chimneys by the metallurgical community or brine channels in sea ice. When these channels are discussed (later), it will be assumed that the scale of the mushy layer and the scale of the channels are greater than the scale of the dendrites.

We begin our analysis with the ideal mushy layer equations, as derived in Lecture 6. The results will therefore describe the physics of the convecting mushy layer but will not agree quantitatively with experiments. Taking

$$\rho = \rho_0[1 - \alpha(T - T_0) + \beta(C - C_0)], \tag{1}$$

$$T \equiv T_L(C) = T_0 - m(C - C_0), \tag{2}$$

$$\theta = \frac{T - T_0}{T_0 - T_B} = \frac{C - C_0}{C_0 - C_B}, \tag{3}$$

$$\Delta T \equiv T_0 - T_B, \tag{4}$$

the ideal mushy layer equations simplify to:

$$\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta = \kappa \nabla^2 \theta + S \frac{\partial \varphi}{\partial t}, \tag{5}$$

$$(1 - \varphi)\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta = -(\xi - \theta)\frac{\partial \varphi}{\partial t}, \tag{6}$$

$$\mathbf{u} = -\frac{\Pi}{\mu}[\nabla p + \rho_0 \beta^* \Delta C \theta \mathbf{g}], \tag{7}$$

where $S = L/(C_P \Delta T)$, $\xi = C_0/(C_E - C_0) = C_0/\Delta C$ and $\beta^* = \beta + m\,\alpha$.

Typically $m\,\alpha < \beta$ which is the reason for denoting β^* as above. For boundary conditions, we have bottom temperature equal to the eutectic temperature, i.e. $T(z = 0) = T_B = T_E$. We also have $T(z = h) = T_0$ and $T(\infty) = T_\infty$. These boundary conditions are for a one-dimensional problem, but are easily generalized to multiple dimensions.

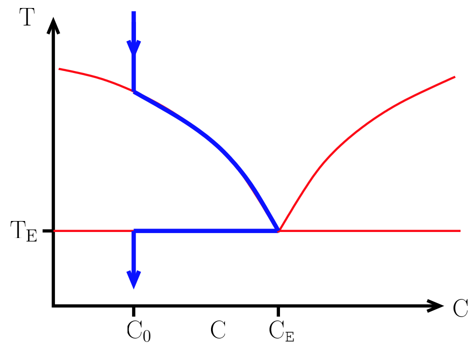

Figure 1: Phase diagram for near-eutectic approximation. In this approximation, ΔC is taken to be much less than C_0.

2 Near-Eutectic Approximation

The near-eutectic approximation (see Fig. 1) is that

$$\xi \gg 1 \tag{8}$$

and

$$S \gg 1 \tag{9}$$

with

$$\frac{S}{\xi} = O(1). \tag{10}$$

Taking this appoximation yields $\varphi \sim 1/\xi \ll 1$. Thus (6) yields

$$-\xi \frac{\partial \varphi}{\partial t} \approx \frac{D\theta}{Dt}. \tag{11}$$

Substituting this result into (5) and defining $\Omega = 1 + S/\xi$ leads to

$$\Omega \frac{D\theta}{Dt} = \kappa \nabla^2 \theta. \tag{12}$$

Equations (7) and (12) are equivalent to those for convection in a passive porous medium.

To solve the equations, we first scale length with h, time with $h^2\Omega/\kappa$, velocity with $\kappa/(h\Omega)$, and pressure with $\beta^* \Delta C \rho_0 g h$. Thus, (12) and (7) become

$$\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta = \nabla^2 \theta \tag{13}$$

and

$$\mathbf{u} = R_m[\nabla p - \theta \hat{\mathbf{k}}], \tag{14}$$

where

$$R_m = \frac{\beta^* \Delta C \rho_0 g \Pi h \Omega}{\kappa \nu} \tag{15}$$

and $\nu = \mu/\rho_0$. Now all variables are dimensionless.

Solving for the basic state (no time dependence or x dependence), we set $\partial/\partial t = \partial/\partial x = 0$ which yields $\theta = -1 + z$ and $\mathbf{u} = 0$. Next we solve the two-dimensional problem with convection solely within the mushy layer and with a planar interface. Introducing a stream function for the 2-dimensional velocity such that, $\mathbf{u} = (\psi_z, -\psi_x)$, and adding a perturbation θ' to the base state temperature field, $\theta = -1 + z + \theta'$, (13) and (14) lead to

$$\nabla^2\theta' = -\psi_x + \frac{\partial\theta'}{\partial t}, \tag{16}$$

$$\nabla^2\psi = -R_m\theta'_x. \tag{17}$$

Assuming no flow at either (planar) interface of the mushy layer, the boundary conditions are

$$\theta'(z = 0) = 0, \tag{18}$$

$$\theta'(z = 1) = 0, \tag{19}$$

$$\psi(z = 0) = 0, \tag{20}$$

$$\psi(z = 1) = 0. \tag{21}$$

One should note that this problem is not realistic since convection in the mushy layer will induce convection in the liquid above so that boundary condition (21) is not satisfied in practical situations. However, the problem is still a useful conceptual problem to solve.

To start the stability analysis, we look for perturbations of the form

$$\theta' = \hat{\theta}(z)e^{i\alpha x+\sigma t}, \tag{22}$$

$$\psi = \hat{\psi}(z)e^{i\alpha x+\sigma t}. \tag{23}$$

Marginal equilibrium then occurs when $\sigma = 0$, or by setting $\frac{\partial}{\partial t} = 0$. Substitution with this condition yields

$$\left(\frac{d}{dz}^2 - \alpha^2\right)\hat{\theta} = i\alpha\hat{\psi} \tag{24}$$

$$\left(\frac{d}{dz}^2 - \alpha^2\right)\hat{\psi} = -i\alpha R_m\hat{\theta} \tag{25}$$

The solution to these coupled ODEs is

$$\hat{\theta} = A_n\sin(n\pi z), \tag{26}$$

$$\hat{\psi} = B_n\sin(n\pi z), \tag{27}$$

$$-R_m = \frac{(n^2\pi^2 + \alpha^2)^2}{\alpha^2}. \tag{28}$$

Plotting $-R_m$ at marginal equilibrium as a function of α we note that the first instability (lowest value of $-R_m$) occurs for $n = 1$, thus the curve

$$-R_m = R_b(\alpha) \equiv \frac{(\pi^2 + \alpha^2)^2}{\alpha^2} \tag{29}$$

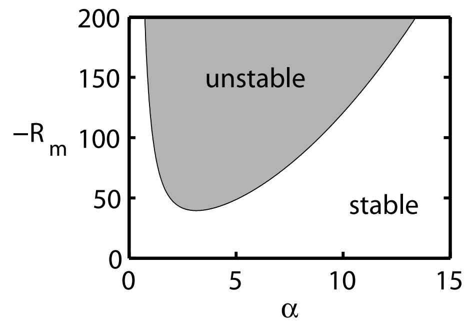

Figure 2: Stability of perturbations with Rayleigh number R_m and wavenumber α.

is the boundary between stability ($-R_m < R_b$) and instability ($-R_m > R_b$) (as a function of α) as shown in Fig. 2. The minimum value of $R_b(\alpha)$ occurs at $R_b(\alpha) = R_c \equiv 4\pi^2$. If $R_m > R_c$ then instability occurs.

Summary of key points

- R_m is a porous-medium Rayleigh number and is proportional to Πh, where Π is permeability of the mushy layer and h is the thickness.
- R_m depends on *compositional* buoyancy but on *thermal* diffusivity. In general, the Rayleigh number is the ratio of buoyancy to dissipation.
- The critical condition is modified by $\Omega = 1 + S/\xi$, so convection is more likely when S is large.

3 Parcel Argument

Consider a parcel in the mushy layer, and hence on the liquidus, that is moved to a different region in the mushy layer where the fluid is warmer and saltier (but still on the liquidus). As is illustrated in Fig. 3, the parcel initially gets warmed to the temperature of its new surroundings; it then dissolves some crystals to increase its salinity and arrive on the liquidus. The dissipation of buoyancy, then, is through a combination of thermal diffusion and dissolution.

Large Stefan number S or small ξ means that there is less phase change per unit temperature change, and hence less dissipation of buoyancy, leading to greater instability. This is also the basic mechanism of channel formation.

From equation (11), we see that there is dissolution where the temperature of a fluid parcel increases. This requires the flow to be larger than the isotherm propagation speed.

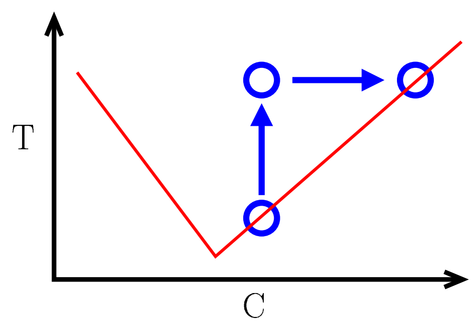

Figure 3: Phase diagram for parcel being moved to warmer region within mushy layer.