Lecture 3: Solutions to Laplace’s Tidal Equations

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1 Introduction

In this lecture we discuss assumptions involved in obtaining Laplace’s Tidal Equations (LTE) from Euler’s equations. We first derive an expression for the solid Earth tides. Solutions of LTE for various boundary conditions are discussed, and an energy equation for tides is presented.

Solutions to the Laplace Tidal equations for a stratified ocean are discussed in §2. We obtain expression for solid earth tide in §3. Different models of dissipation are examined in §4. Boundary conditions for LTE’s are discussed in §5. In §6 the energy equation for LTE’s is derived.

2 The Laplace Tidal equations for a Stratified ocean

To obtain the LTE for stratified ocean we assume pressure is hydrostatic and seek separable solutions of the form

\[ u(x, y, z, t) = U(x, y, t)F_u(z), \]
\[ w(x, y, z, t) = W(x, y, t)F_w(z), \]
\[ p(x, y, z, t) = Z(x, y, t)F_p(z). \]

The LTE for stratified ocean are

\[ U_t - fV = -gZ_x , \]  
\[ V_t - fU = -gZ_y , \]  
\[ Z_t + D_n(U_x + V_y) = 0 , \]  
\[ F_{wzz} + \frac{N^2}{gD_n}F_w = 0 , \]

where \( n \) is an index for the normal modes in the ocean. These equations are the constant depth LTE but where \( D_n \) is the equivalent depth of each mode, and \( D_n \neq D_n \), rather

\[ D_n = \frac{[\int_{-D_n}^{0} N(z')dz']^2}{g n^2 \pi^2} \]
and $N(z)$ is the buoyancy frequency. For the zeroth mode or $n = 0$, $D_s \approx D$. The eigenvalue problem in (4) can be solved to determine the $D_n$ using the following boundary conditions

\begin{align*}
F_w &= 0 \quad \text{at} \quad Z = -D_s, \\
F_w - D_n F_{w_z} &= 0 \quad \text{at} \quad Z = 0.
\end{align*}

2.1 Barotropic Solution ($n = 0$)

The normal mode equations described above and indexed by $n$ an integer can be solved for specific modes. The zeroth order mode is given by $n = 0$ and is also called the barotropic or external mode. It is characterized by a solution which is depth independent. Below we will solve for the barotropic mode without rotation and no variations in the y direction ($f = 0$ and $\partial/\partial y = 0$). If we assume a plane wave solution, solving (1) - (4) gives,

\begin{align*}
Z_0 &= ae^{-ist+ikx}, \\
U_0 &= a \left( \frac{gk}{\sigma} \right) e^{-ist+ikx}, \\
\sigma &= \pm k\sqrt{gD_s}.
\end{align*}

Figure 1 shows the barotropic solution for velocity. For the semidiurnal tidal frequency, the phase velocity of the zeroth mode wave, given by $c_0 = \sqrt{gD_s} = \sigma/\kappa = 200$ m/s. Since this speed is also given by $\lambda/T$, where $\lambda$ is the horizontal wavelength of the wave (distance from wave crest to wave crest), then $\lambda = 8640$ km. This wave is very fast and very long. For the case of no rotation this wave is dispersionless, but not when $f \neq 0$.

2.2 Baroclinic Solution, Mode 1 ($n = 1$)

The first baroclinic mode, indexed by $n = 1$ is also called the first internal mode. The rest of the modes for $n > 1$ are also internal modes and have more variation in depth. We can solve (1) - (4) for $n = 1$ without rotation and with no variations in y ($f = 0$ and $\partial/\partial y = 0$) giving the first internal mode,

\begin{align*}
Z_n &= ae^{-ist+ikx}, \\
F_{U_1} &= \cos \left( \frac{\pi(z + D_s)}{D_s} \right), \\
F_{W_1} &= \sin \left( \frac{\pi(z + D_s)}{D_s} \right), \\
\sigma &= \pm k\sqrt{gD_1}.
\end{align*}

The mode one solution is shown in figure 1. For the baroclinic modes, the phase velocity and horizontal wavelength are given by

\begin{align*}
c_n &= \frac{N_0 D_s}{n\pi} = \frac{(1.45 \times 10^{-3})(4000m)}{n\pi} = \frac{1.85 \text{ m/s}}{n}, \\
\lambda_{\text{tidal}} &= \frac{80 \text{ km}}{n}.
\end{align*}
So the lower modes travel more quickly, and have larger horizontal scales. The $M_2$ internal tide is found primarily as a mode 1 tide throughout the world’s oceans, while higher order modes tend to dissipate nearer their source. Due to recent (last decade) improvements, the $M_2$ internal tide can now be seen by satellite altimetry. The ocean surface displacement due to internal tides is given by

$$\frac{w_{\text{free surface}}}{w_{\text{interior maximum}}} \approx \frac{N_0^2 D_x}{g n \pi} \approx 3 \times 10^{-4},$$

which means that for an internal tide displacement of isopycnals on the order of 100m (which is quite large but not impossible), the surface expression would be about 30cm, easily resolved by satellite altimetry measurements which have accuracy on the order of a few centimeters.

![Figure 1](image.png)

Figure 1: (a) Barotropic or depth independent solution for $u$ velocity. Wave amplitude is greatly exaggerated. (b) Mode 1 solution for $u$ and $w$ velocity.

### 2.3 Numerical Solution to LTE

The complete problem that we would like to solve numerically to estimate the tides are the stratified linear equations,

$$u_t - fv = -\frac{p_x}{\rho_0} + \Gamma_x,$$  \hspace{1cm} (5)

$$v_t + fu = -\frac{p_y}{\rho_0} + \Gamma_y,$$  \hspace{1cm} (6)

$$\zeta_t + (uD)_x + (vD)_y = 0,$$  \hspace{1cm} (7)

where $\Gamma$ is full tide generating potential and $D(x, y)$ is the bottom topography. The domain is defined by the coasts of continents, ocean bottom and free surface. However, in order to solve this system of equations one must resolve short horizontal scales due to bottom topography where the bottom boundary condition on $w$ is $w = -\mathbf{u} \cdot \nabla D$. Very few current modes are capable of this, though some have begun to resolve mode 1 in their simulations.
Instead the Laplace tidal equations for $u$ and $v$ may be substituted, of (5)-(7).

\begin{align*}
    u_t - fv &= -g\zeta_x + \Gamma_x, \\
    v_t + fu &= -g\zeta_y + \Gamma_y,
\end{align*}

(8) (9)

where $\zeta$ is the free surface and a smoother bottom topography is substituted,

\[ w = -\mathbf{u} \cdot \nabla D_{\text{smooth}}. \]

(10)

However, because $\zeta$ is no longer a function of $z$ while $p$ in (5)-(7) was, we cannot determine $u(z)$ and (8)-(9) will only give the barotropic solution.

In the literature, the TGP is usually neglected and instead the barotropic tides and stratification are specified, which allows the simplification of (5)-(7) as

\begin{align*}
    u_t - fv &= -\frac{px}{\rho_0}, \\
    v_t + fu &= -\frac{py}{\rho_0}.
\end{align*}

(11) (12)

From this, the internal tides result from a single scattering of the barotropic tide by bottom relief $w_{\text{int}} = u_B \cdot \nabla D(x, y)$. In particular, if we decomposed $D$ into low- and high-passed components,

\[ D(x, y) = D_{lo}(x, y) + D_{hi}(x, y). \]

(13)

Then the $\zeta$ equation and bottom boundary conditions become

\[ \zeta_t + \nabla \cdot (u_B D_{lo}(x, y)) = 0, \]

\[ w_{\text{int}} = u_B \cdot \nabla D_{hi}(x, y). \]

And the internal tide results from the bottom topography. However, this neglects multiple scattering from the topography and does not apply when the bottom slope is greater than the characteristic slope of internal waves. Currently, numerical models like the Princeton Ocean Model (POM) solve the (5)-(7). An example of numerically solved tides is shown in figure 2. In this paper, all tidal constituents were solved for using a hydrodynamic model and data assimilation from tide gauges and altimetry [1].

\section{Solid Earth Tide}

It has long been known that Earth’s crust yields elastically to the tidal forces of the moon and sun. If we consider earth to be an incompressible elastic solid, then we can write equations for the deformation of the Earth as the following

\begin{align*}
    -p_x + \mu \nabla^2 u &= 0, \\
    -p_z + \mu \nabla^2 w &= 0.
\end{align*}

(14) (15)

where, $p$ is pressure, $(u, v)$ are the velocity and $\mu$ is viscosity of earth (see figure 3). Using the following boundary conditions,
Figure 2: Cotidal map of the $M_2$ component. Coamplitude lines are drawn following the scaling indicated below the map. Units are in centimeters. Cophase lines are drawn with an interval of $30^\circ$, with the $0^\circ$ phase as a larger drawing, referred to the passage of the astronomical forcing at Greenwich meridian [1].

\[
\begin{align*}
\text{Figure 3: Solid earth tide, } \zeta \text{ is geocentric surface tide and } \delta \text{ is geocentric solid earth tide.}
\end{align*}
\]

\[
\begin{align*}
&u, w \to 0 \text{ as } z \to -\infty, \\
&\tau_{xz} = 2\mu(u_z + w_x) = 0 \text{ at } z = 0, \\
&\tau_{zz} = -p + 2\mu w_x = \text{load} \equiv -\rho_w g a e^{ikx},
\end{align*}
\]

where term $\rho_w g a e^{ikx}$ gives the loading on earth surface due to ocean tide, $a$ is the tidal amplitude and $k$ is its horizontal wavenumber, we solve (14) and (15) with the above b.c.’s for a load of $\rho_w g a e^{ikx}$ to get an Earth surface wave displacement of

\[
\begin{align*}
h \rho_w g a e^{ikx},
\end{align*}
\]

where $h$ is called “Love Number”. In this case, $h = \frac{1}{2\pi k}$. 
We can then write the tide generating potential in spherical harmonic \((n)\) decomposition including the solid Earth tide as,

\[
\Gamma_n = U_n + k_n U_n + q_n \alpha_n \zeta_{on} + k'_n q_n \alpha_n \zeta_{on},
\]

(17)

where \(U_n\) is obtained from astronomy, \(k_n U_n\) is the earth yielding to astronomical potential, \(q_n \alpha_n \zeta_{on}\) is potential of tidal shell and \(k'_n q_n \alpha_n \zeta_{on}\) is earth yielding to tidal potential.

Proceeding as above, solid earth tide is given as,

\[
\delta_n = h_n U_n + h'_n \alpha_n \zeta_{on}.
\]

(18)

\(k_n, k'_n, h_n\) and \(h'_n\) are all “Love numbers” similar to \(h\) in (16). From (17) and (18) we find

\[
\left( \frac{\Gamma}{g} - \delta \right)_n = \left( 1 + \frac{k_n - h_n}{g} \right) U_n + (1 + k'_n - h'_n) \alpha_n \zeta_{on}.
\]

(19)

And upon summing up this series we get Farrell Green’s function [2]

\[
\sum_n (1 + k'_n - h'_n) \alpha_n \zeta_{on} = \int \int d\theta' d\phi' G_F(\theta', \phi' | \theta, \phi) \zeta_0(\theta', \phi').
\]

(20)

Now taking into account the solid Earth tide, we can rewrite Euler’s equations from Lecture 2 as

\[
\zeta_0 = \zeta - \delta,
\]

(21)

\[
u_t - (2\Omega \sin \theta) v = -\frac{g(\zeta_0 - (\Gamma/g - \delta))}{a \cos \theta} + \frac{F^\phi}{\rho D},
\]

(22)

\[
v_t + (2\Omega \sin \theta) u = -\frac{g(\zeta_0 - (\Gamma/g - \delta))}{a} + \frac{F^\theta}{\rho D},
\]

(23)

\[
(\zeta_t - \delta_t) + \frac{1}{a \cos \theta} [(uD)_\phi + (vD \cos \theta)_{\theta}] = -\sum_n \left( \frac{1 + k_n - h_n}{g} \right) U_n
\]

\[
- \int \int d\theta' d\phi' G_F(\theta', \phi' | \theta, \phi) \zeta_0(\theta', \phi'),
\]

(24)

where \(U\) is mostly \(U_2\), \(k_2 \approx 0.29\) and \(h_2 \approx 0.59\).

4 Dissipation Models

It is not easy to estimate the dissipation terms \((F^\theta, F^\phi)\) in (22) and (23). This dissipation is mainly due to bottom drag and internal tides. If we model it as bottom drag, we get

\[
F = -\rho C_D |u| \dot{u},
\]

(25)

where \(C_D \approx 0.0025\) known from direct measurement in shallow water. The direct effect of (25) is that most of dissipation is limited to shallow seas where \(u\) and \(C_D\) are large. However, global tidal computations are mostly confined to deep-water zones for practical reasons (shallow water tides require much finer grid-spacing). So dissipation can only be properly represented by radiation of energy out of the model into bounding seas.

There are two main empirical models used to get an expression for dissipation in deep oceans. These are,
Jayne & Laurent In their runs of the Hallberg Isopycnal Model (HIM), the dissipation term is modeled as,

\[ F = -\frac{1}{2} \rho k h^2 N_b u. \] (26)

This model is based on assumption that dissipation occurs due to barotropic tide scattering into internal tides due to the rough bottom topography. \( h \) represents the height of bottom topography whose dominant horizontal wavenumber is \( k \) (see figure 4).

Arbic In his runs of the HIM for the dissipation term we have,

\[ F = -\rho'_W \nabla h = \rho (\nabla \chi \cdot \mathbf{u}) \nabla h, \] (27)

where,

\[ \chi = \frac{N_b \sqrt{\sigma^2 - f^2}}{2\pi \sigma} \int \int \frac{h(x')}{|x - x'|} dx dy' \] (28)

for tide of frequency \( \sigma \). \( N_b \) is buoyancy frequency, \( f \) is the Coriolis parameter and \( h(x) \) is bottom topography. Arbic’s model is based on the assumption that tidal dissipation can be calculated by finding the pressure drop in tidal currents across topographic features at the bottom.

5 Boundary Conditions

Laplace tidal equations have never been solved well enough so as to remove tides from altimetry without data assimilation. Different methods use different boundary conditions for solution of LTE at numerical coast. Some of the main boundary conditions in use are,

1. \( \mathbf{u} \cdot \mathbf{h} = 0 \) at the numerical coast.

   This is the most commonly used boundary condition. This boundary condition represents a no-energy-flux coast. It is important to have correct information regarding dissipation if this boundary is used since all energy must be dissipated within the system. Numerical schemes need to resolve \( -\rho C_D \mathbf{u} |\mathbf{u}| \) well which requires high resolution in shallow waters.

Figure 4: Jayne & Laurent model, dissipation due to barotropic tide scattering into internal tides over rough topography.
2. $\zeta = \zeta_{\text{obs}}$ at the numerical coast.
   This boundary condition allows an energy flux $(\rho g D |u| \cdot \hat{h}\zeta)$ through the coast. The scheme is less sensitive to the details of the dissipation model used, and is less sensitive to the discretization used. However, this system can still respond resonantly. Another problem with this scheme is that observed tidal data is not easily available along all coasts.

3. $\zeta = c (\mathbf{u} \cdot \hat{h})$ at the numerical coast.
   This boundary allows energy to be dissipated at coast. In this boundary the parameter $c$ can be adjusted so as to get results to match the observed tidal results. This leads to an energy flux of $< \rho g D c \zeta^2 >$ flowing out of the coast.

6 Energetics

If we ignore the solid Earth tide, we can derive equations of energy and perhaps estimate dissipation due to the tides as a residual. Starting with Laplace’s tidal equations,

\[
\begin{align*}
    u_t - f v &= -g(\zeta - \Gamma / g)_x + F^x / \rho D \times \rho u D \\
    v_t + f u &= -g(\zeta - \Gamma / g)_y + F^y / \rho D \times \rho v D \\
    \zeta_t + (u D)_x + (v D)_y &= 0
\end{align*}
\]

Then multiplying by the terms at right, adding the three equations together and assuming that $\rho$, $g$ and $D$ are constant, we arrive at

\[
\frac{1}{2} \rho D (u^2 + v^2)_t + \frac{1}{g} \rho g (\zeta^2)_t + \nabla \cdot (\rho g \zeta u D) = \rho \zeta \Gamma + \nabla \cdot (\rho u D \Gamma) + u \cdot F
\]  

(29)


\[
\begin{array}{ccc}
    \text{Fluid crossing} & \text{Fluid crossing} & \text{Work by} \\
    \text{vertically} & \text{horizontally} & \text{dissipative forces}
\end{array}
\]

where $KE_t$ is the time derivative of kinetic energy, $PE_t$ is the time derivative of potential energy, and $Eflux$ is energy flux.

Energy Averaged Over One Tidal Period, Integrated Over the Ocean

It is convenient to consider the energy as averaged over one tidal period. For a periodic tide, let $\langle \cdot \rangle$ denote the average over one period. This will simplify the above equations, since

\[
\langle KE_t \rangle = \langle PE_t \rangle = 0.
\]  

(31)

Then from (29) we are left with

\[
\nabla \cdot \langle P \rangle = \langle W_t \rangle + \langle u \cdot F \rangle,
\]  

(32)

where $P$ is energy flux and $W_t$ is the working by potential vertical and horizontal forces.

Now if we reconsider the case of the basin with no flow through its boundaries ($\vec{w} \cdot \hat{n} = 0$), then we further have that $\int \nabla \cdot \langle P \rangle \, dxdy = 0$ since there can be no
net energy flux into or out of the basin. Then we can integrate the remaining terms in the energy equation over the ocean basin and find that

$$\int < W_t > dx dy = \int_{\text{ocean}} \rho \zeta_0 \Gamma dx dy = - \int_{\text{ocean}} < u \cdot F > dx dy. \quad (33)$$

One caveat is that the solid Earth tide is not dissipation free, i.e. the Love numbers are complex, but this equation is true provided that they are real. Now, since $$\int < u \cdot F > dx dy$$ is balanced by working of fluid moving up and down. This fluid movement $$\zeta$$ can be measured from global altimetry, giving an estimate if dissipation of energy due to the tides.

### 6.1 Including the solid earth tide

If we now include the solid Earth tide, then our third equation becomes

$$\begin{align*}
(\zeta - \delta)_t + \nabla \cdot \vec{u} \vec{D} &= 0 \\
\end{align*} \quad (34)$$

and if we follow the same procedure as before we have

$$\begin{align*}
KE_t &= -\nabla \cdot (\rho \vec{D} \vec{u} (\zeta - \Gamma / g)) + \rho g(\zeta - \Gamma / g) \nabla \cdot \vec{u} \vec{D} + u \cdot F \\
&= -\nabla \cdot (\rho \vec{D} \vec{u} (\zeta - \Gamma / g)) - \rho g(\zeta - \Gamma / g)(\zeta_t - \delta_t) + \vec{u} \cdot F \\
&= \nabla \cdot \rho \vec{D} \vec{u} \Gamma + \rho (\zeta - \delta)_t \Gamma - \nabla \cdot \rho \vec{D} \vec{u} \zeta - \rho g(\zeta - \delta)_t + u \cdot F.
\end{align*}$$

Similarly for potential energy,

$$\begin{align*}
PE = \int^{\zeta}_{-\delta - \delta} \rho g dz &= \frac{1}{2} \rho g (\zeta^2 - (-\delta - \delta)^2), \\
PE_t &= \rho g (\delta_t - \zeta_t - D \zeta_t).
\end{align*}$$

Adding these together we find that

$$\begin{align*}
KE_t + PE_t + \nabla \cdot (\rho g \vec{u} \vec{D}) &= \nabla \cdot \rho \vec{D} \vec{u} \Gamma + \rho (\zeta - \delta)_t \Gamma + u \cdot F \\
&- \rho g(\zeta_t - \delta_t) + \rho g(\zeta_t - \delta \delta_t + D \delta_t) \\
&= \nabla \cdot \rho \vec{D} \vec{u} \Gamma + \rho (\zeta - \delta)_t \Gamma + u \cdot F + \rho g(\zeta - \delta + D \delta_t).
\end{align*}$$

Then if we consider the observed tide only, $$\zeta_0 = \zeta - \delta$$, the difference between the ocean tides and the solid earth tide, we have the energy equation.

$$\begin{align*}
KE_t + PE_t + \nabla \cdot \vec{P} = W_t + u \cdot F, \quad (35)
\end{align*}$$

with

$$\begin{align*}
KE &= \frac{1}{2} \rho D (u^2 + v^2), \quad (36) \\
PE &= \frac{1}{2} \rho g (\zeta_0^2 + 2 \zeta_0 \delta + 2 \delta D), \quad (37) \\
\vec{P} &= \rho g \vec{D} \vec{u} (\zeta_0 + \delta), \quad (38) \\
W_t &= \rho \zeta_0 \Gamma + \rho \nabla \cdot u \vec{D} \Gamma + \rho g (\zeta_0 + D) \delta_t. \quad (39)
\end{align*}$$
Energy Averaged Over One Tidal Period, Integrated Over the Ocean

If we again average over one tidal period, then

$$\nabla \cdot < P > = < W_t > + < u \cdot F >$$  \hspace{1cm} (40)

Given altimeter data $\zeta_1$ it may be possible to map $< u \cdot F >$ [3].

If we further assume that the tides are periodic as $(e^{-i\omega t})$, then noting that in the equation for $W_t$ that $\int_{\text{ocean}} \rho \nabla \cdot u D\Gamma = 0$ and $< \rho g D \delta_t > = 0$,

$$\int_{\text{ocean}} < W_t > dxdy = \frac{1}{2} Re \left\{ \int t_0 - \sigma \rho \zeta_0 \Gamma^* + i \sigma g \rho \zeta_0 \delta^* \right\},$$

where $(\cdot)^*$ is the complex conjugate. Using the Love number decomposition from §3,

$$\int_{\text{ocean}} < W_t > dxdy = \frac{1}{2} Re \left[ \sum_n -i \sigma \rho \zeta_{0n} \left( (1 + k_n) u_n^* + g \alpha_n \zeta_{0n} + k_n^* g \alpha_n \zeta_{0n} \right) \right. $$

$$+ \left. i \sigma \rho \zeta_{0n} (h_n u_n^* + \delta h_n^* \alpha_n \zeta_{0n}) \right]$$

$$= \frac{1}{2} Re \sum_n \int -i \sigma \rho (1 + k_n - h_n) \zeta_{0n} u_n^*.$$  \hspace{1cm} (41)

This last equation is true provided that the solid earth tide is dissipation-less, that is to say, that the Love numbers are real. Now, since again $\int_{\text{ocean}} \nabla \cdot < P > dxdy = 0$,

$$\int_{\text{ocean}} < u \cdot F > dxdy = \int_{\text{ocean}} < W_t > dxdy = \frac{1}{2} Re \sum_n \int -i \sigma \rho (1 + k_n - h_n) \zeta_{0n} u_n^*  \hspace{1cm} (42)$$

and we can estimate the dissipation of energy due to the tides if we know the observed tides, $\zeta_0$.

Notes by Vineet Birman and Eleanor Williams Frajka

References


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<td>Pekeris et al. (1969)</td>
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<td>Tiron et al. (1967)</td>
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<td>Platzman (1972)</td>
<td>Normal modes for major basins, especially Atlantic</td>
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<td>Zero</td>
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<td>Munk et al. (1970)</td>
<td>Normal mode representation of California coastal tides</td>
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<td>Cartwright (1971)</td>
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Table 1: Summary of Large-Scale Tidal Models.