Wave–mean-flow interaction in Oldroyd-B fluid

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1 Introduction

The effects of waves on the mean-flow have been extensively studied, most notably in the studies of oceanic and atmospheric waves. But waves in non-Newtonian fluids have not received much attention. I will be focusing here on the wave–mean-flow interaction in Oldroyd-B fluid. This is done in a very simple flow profile to study clearly various phenomena arising because of elasticity.

I will start by studying the linear equations in section Sec. 2. The equations for mean-flow response are obtained after introducing zonal averaging in section Sec. 3. I discuss and apply the Generalized Lagrangian Mean theory in section Sec. 4. The spin-up and spin-down problem, discussed in section Sec. 5, illustrates some of the peculiar features of the mean-flow response. I conclude with a few remarks and indicate some directions for further studies.

2 Linear theory

The Oldroyd-B model for an incompressible fluid is given by:

$$\nabla \cdot \mathbf{u} = 0\,, \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u} + G \nabla \cdot \mathbf{A}\,, \tag{2}$$

$$\frac{\partial \mathbf{A}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{A} - (\nabla \mathbf{u})^T \cdot \mathbf{A} - \mathbf{A} \cdot \nabla \mathbf{u} = -\frac{1}{\tau}(\mathbf{A} - \mathbf{I})\,. \tag{3}$$

I have set the constant fluid density $\rho = 1$. The momentum equation contains the divergence of the polymeric stress $G\mathbf{A}$. This extra stress simply advects with the flow, as given by the "upper convected derivative" [left hand side of (3)], but it also relaxes to $\mathbf{I}$ with a time constant τ.

multiplying the momentum equation (2) by $\mathbf{u}\cdot$ we can get the equation for energy:

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right) \left(\frac{1}{2}\mathbf{u}^2 + \frac{G}{2}\mathrm{tr}\ A\right)$$
$$= \nabla \cdot (-\mathbf{u}p + \nu \nabla \mathbf{u} \cdot \mathbf{u} + G\mathbf{u} \cdot \mathbf{A}) - \nu \nabla \mathbf{u} : \nabla \mathbf{u} - \frac{G}{2\tau}\left(\mathrm{tr}\ A - 3\right)\,. \tag{4}$$

Thus we see that in addition to viscosity, the relaxation of polymeric stress also dissipates energy. In most of the following, I will consider the "ideal" limit of this model: an inviscid, relaxation-less Oldroyd-B fluid with $\nu = 0$ and $1/\tau = 0$. In this limit, the energy is conserved.

I consider the flow in an semi-infinite two dimensional domain $D = \{(x, y)| -\infty < x < \infty, h(x) <= y < \infty\}$. We can satisfy two boundary conditions at the boundary $y = h(x)$: the free-slip condition $\mathbf{u} \cdot \hat{\mathbf{n}} = 0$, and the condition of no tangential stress $\hat{\mathbf{n}} \times (\mathbf{A} \cdot \hat{\mathbf{n}}) = 0$. Here $\hat{\mathbf{n}}$ is the normal to the lower boundary: $\hat{\mathbf{n}} = (-h_x(x), 1)$.

I will study waves on following one dimensional constant flow profile, thus avoiding problems of critical layers:

$$\mathbf{u} = (U, 0)\,, \qquad p = p_0\,, \qquad \mathbf{A} = \mathbf{I}\,, \qquad \text{and} \qquad h(x) = 0\,. \tag{5}$$

In the absence of relaxation, the stress $\mathbf{A}$ can be any constant matrix, not necessarily $\mathbf{I}$. But the above choice was made with the following in mind: The qualitative features of the wave–mean-flow interaction do not change by assuming it to be $\mathbf{I}$; Also, one of the extensions of this problem, to be studied later, is the flow in presence of relaxation when the stress $\mathbf{A}$ for the background flow must be $\mathbf{I}$.

Substituting

$$\mathbf{u} = (U, 0) + \mathbf{u}'\,, \qquad p = p_0 + p'\,, \qquad \text{and} \qquad \mathbf{A} = \mathbf{I} + \mathbf{A}'\,, \tag{6}$$

in (1)-(3), denoting $D_t := \partial/\partial t + U\partial/\partial x$, and keeping terms linear in perturbed quantities, we get the following linear equations:

$$\nabla \cdot \mathbf{u}' = 0\,, \tag{7}$$

$$D_t \mathbf{u}' = -\nabla p' + G\nabla \cdot \mathbf{A}'\,, \tag{8}$$

$$D_t \mathbf{A}' = \left(\nabla \mathbf{u}'\right)^T + \nabla \mathbf{u}'\,. \tag{9}$$

The boundary for the linear problem is chosen to be $h(x) = h_0 \cos(kx)$ with a small amplitude h_0, i.e., $a := h_0 k \ll 1$ is the small parameter. The energy conservation equation for the linear problem is the following:

$$D_t \left(\frac{1}{2}\mathbf{u}'^2 + \frac{G}{4}\text{tr }\mathbf{A}'^2 \right) = \nabla \cdot \left(-\mathbf{u}' p' + G\mathbf{u}'\mathbf{A}' \right)\,. \tag{10}$$

I will introduce the particle displacement associated with the perturbation flow $\mathbf{u}'$ as $D_t \boldsymbol{\xi}' = \mathbf{u}'$. Then we can explicitly integrate (9) to get

$$\mathbf{A}' = (\nabla \boldsymbol{\xi}')^T + \nabla \boldsymbol{\xi}'\,. \tag{11}$$

Using the incompressibility equation $\nabla \cdot \boldsymbol{\xi}' = \mathbf{0}$, we get

$$\nabla \cdot \mathbf{A}' = \nabla^2 \boldsymbol{\xi}' + \boldsymbol{\nabla}(\nabla \cdot \boldsymbol{\xi}') = \nabla^2 \boldsymbol{\xi}'. \tag{12}$$

This reduces the momentum equation (8) to

$$D_t \mathbf{u}' = -\nabla p' + G\nabla^2 \boldsymbol{\xi}'\,. \tag{13}$$

By taking the divergence of the above equation, and using incompressibility, we see an important consequence that pressure is an harmonic function:

$$\nabla^2 p' = 0 \tag{14}$$

The relation (11) is reminiscent of the stress-strain relation for a solid. In fact, we will see later that the vorticity waves for this linear model are the same as elastic waves in an incompressible linear elastic solid. The analogy fails when I consider the potential flow and satisfy the boundary condition on tangential stress: a solid can support tangential stresses at the boundary and these must be specified to solve the problem, while I impose the condition that the tangential stress is zero for this ideal Oldroyd-B fluid. Thus the only difference between a solid and this ideal limit of Oldroyd-B fluid is in the boundary conditions.

We can find two kinds of waves from the linear equations. I begin by considering the vorticity waves. With vorticity defined as $q' = v'_x - u'_y$, we get

$$\begin{aligned} D_t^2 q' &= D_t \left(D_t v'_x - D_t u'_y \right), \\ &= D_t \left(\left(-p_{xy} + G\nabla^2 \eta'_x \right) - \left(-p_{xy} + G\nabla^2 \xi'_x \right) \right), \\ &= G\nabla^2 \left(D_t \left(\eta'_x - \xi'_x \right) \right), \end{aligned}$$

which gives the vorticity wave equation:

$$(D_t^2 - G\nabla^2) q' = 0\,. \tag{15}$$

The dispersion relation for the vorticity waves [with $q' \sim \exp(i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega t)$] is

$$\omega = Uk \pm \sqrt{G}|\boldsymbol{k}| =: Uk + \hat{\omega}\,. \tag{16}$$

Here, $\hat{\omega}$ is the *intrinsic* frequency of the waves ,i.e., frequency in the frame moving with the background flow, whereas ω is the frequency with respect to the boundary. The magnitudes of intrinsic group and phase velocities are equal and are given by

$$v_p = v_g = \pm\sqrt{G}\,. \tag{17}$$

Stationary vorticity waves have $\omega = 0$. Choosing $U > 0$ and $k > 0$, we see that we must choose the lower sign in (16). Then solving for l, we get

$$l = \pm k\sqrt{\frac{U^2}{G} - 1}\,. \tag{18}$$

This shows that for $U^2 > G$ the waves are propagating while for $U^2 < G$ they are evanescent. This is also seen by writing (15) as

$$[(U^2/G - 1)\partial_x^2 - \partial_y^2]\mathbf{q}' = 0\,. \tag{19}$$

I note that this equation is elliptic for $U^2 < G$ and parabolic for $U^2 = G$, but is hyperbolic otherwise.

I choose the lower sign in (18) so that these are outgoing waves (i.e., the y-component of $v_p\boldsymbol{k}/|\boldsymbol{k}|$ is positive). These are pressure-less waves $p'=0$. All the other fields are given by

$$\xi' = -\frac{G\sqrt{U^2/G-1}}{kU^2}\cos(kx+ly)\,, \qquad \eta' = -\frac{G}{kU^2}\cos(kx+ly)\,, \tag{20}$$

$$u' = \frac{G\sqrt{U^2/G-1}}{U}\sin(kx+ly)\,, \qquad v' = \frac{G}{U}\sin(kx+ly)\,, \tag{21}$$

$$A'_{xx} = -A'_{yy} = \frac{2G\sqrt{U^2/G-1}}{U^2}\sin(kx+ly)\,, \qquad A'_{xy} = \frac{2G-U^2}{U^2}\sin(kx+ly)\,. \tag{22}$$

The vorticity waves alone cannot satisfy both the free-slip and tangential stress-free boundary conditions. Thus I look for the other solution of the linear problem, i.e., potential flow. Assuming $\mathbf{u}'=\nabla\phi'$, i.e. $q'=0$, we get from the continuity equation that ϕ' is a harmonic function $\nabla^2\phi'=0$. Requiring that ϕ' oscillates in x, remains bounded as $y\to\infty$, and looking for stationary solutions ($\partial/\partial t=0$), we get

$$\phi' = \Re\frac{U}{k}ie^{ikx-ky} = -\frac{U}{k}\sin(kx)e^{-ky}\,. \tag{23}$$

Using $\mathbf{u}'=\nabla\phi'$, $D_t\boldsymbol{\xi}'=(\partial/\partial t+U\partial/\partial x)\boldsymbol{\xi}'=U\partial\boldsymbol{\xi}'/\partial x=\mathbf{u}'$, and $\mathbf{A}'=(\nabla\boldsymbol{\xi}')^T+\nabla\boldsymbol{\xi}'$ gives all the quantities for the potential flow:

$$\xi' = -\frac{1}{k}\sin(kx)e^{-ky}\,, \qquad \eta' = -\frac{1}{k}\cos(kx)e^{-ky}\,, \tag{24}$$

$$u' = -U\cos(kx)e^{-ky}\,, \qquad v' = U\sin(kx)e^{-ky}\,, \tag{25}$$

$$A'_{xx} = -A'_{yy} = -2\cos(kx)e^{-ky}\,, \qquad A'_{xy} = 2\sin(kx)e^{-ky}\,. \tag{26}$$

The pressure for the potential flow is not zero but decays exponentially: $p'=U^2\cos(kx)e^{-ky}$.

Both the above solutions are written so that the y-particle displacement is in phase with the boundary $h(x)$. I write the total solution as $\xi'=\alpha\xi'_{(1)}+\beta\xi'_{(2)}$, where $\xi'_{(1)}$ and $\xi'_{(1)}$ are respectively the vorticity wave and potential flow solutions. The constants α and β can be found from the two boundary conditions as follows:

$$\begin{aligned}
\mathbf{u}\cdot\hat{\mathbf{n}}=0\,,\Rightarrow\quad & v'|_{y=0}=Uh_x(x)\,, &\Rightarrow\quad & G\,\alpha+U^2\beta &= -aU^2\,.\\
\hat{\mathbf{n}}\times(\mathbf{A}\cdot\hat{\mathbf{n}})=0\,,\Rightarrow\quad & A'_{xy}|_{y=0}=0\,, &\Rightarrow\quad & (2G-U^2)\,\alpha+2U^2\beta &= 0\,.
\end{aligned}$$

Solving the last two equations, we get

$$\alpha=-2a, \qquad \text{and} \qquad \beta=a\left(\frac{2G}{U^2}-1\right). \tag{27}$$

Thus, we see that $\beta=0$ for $U^2=2G$, which can also be seen directly from (22), because $A'_{xy}=0$ in that case and the boundary condition is satisfied with the vorticity waves alone. This specific velocity will be important again when we later consider drag.

Briefly going back to the full set of Oldroyd-B equations (1)-(3), I will get the dispersion relation for vorticity waves with relaxation and dissipation. The linear equations in that case are

$$\nabla \cdot \mathbf{u}' = 0\,, \tag{28}$$

$$D_t \mathbf{u}' = -\nabla p' + \nu \nabla^2 \mathbf{u}' + G \nabla \cdot \mathbf{A}'\,, \tag{29}$$

$$D_t \mathbf{A}' = \left(\nabla \mathbf{u}'\right)^T + \nabla \mathbf{u}' - \frac{1}{\tau} \mathbf{A}'\,. \tag{30}$$

Then the equations for vorticity $\boldsymbol{q'} = \nabla \times \mathbf{u}$ and $\boldsymbol{\Omega'} := \nabla \times (\nabla \cdot \mathbf{A}')$ become

$$D_t \boldsymbol{q'} = -\frac{1}{\tau} \boldsymbol{q'} + \nabla^2 \boldsymbol{\Omega'}\,, \tag{31}$$

$$D_t \boldsymbol{\Omega'} = \nu \nabla^2 \boldsymbol{\Omega'} + G \boldsymbol{q'}\,. \tag{32}$$

These equations give the following dispersion relation:

$$\omega = Uk - \frac{i}{2}\left(\frac{1}{\tau} + \nu |\boldsymbol{k}|^2\right) \pm \sqrt{G} |\boldsymbol{k}| \sqrt{1 - \frac{(\nu \tau |\boldsymbol{k}|^2 - 1)^2}{4 G \tau^2 |\boldsymbol{k}|^2}}\,. \tag{33}$$

As another aside, if the background $\mathbf{A}$ is not equal to $\mathbf{I}$ but some constant symmetric matrix $\mathbf{M}$, then the dispersion relation is given by

$$\hat{\omega} = \sqrt{G} \sqrt{\boldsymbol{k} \cdot \mathbf{M} \cdot \boldsymbol{k}}\,. \tag{34}$$

This is very much like the dispersion relation for Alfvén waves in magnetohydrodynamic flows. This kind of analogy between non-Newtonian fluid flow and magnetohydrodynamic has been studied in different context in [5] and exploring it in greater details will be interesting.

3 Zonal averaging and small amplitude expansion

We will be interested in the effect of the waves on the mean-flow. To study this, we introduce the concept of *zonal averaging*, which is defined by

$$\overline{f} := \frac{1}{L} \int_0^L f(x, y, t) dx\,, \tag{35}$$

for any function which is periodic in x with period L. I will take L to be the wavelength $2\pi/k$ of the boundary. The *disturbance part* is defined as $f' := f - \overline{f}$. This is an exact decomposition without any assumption about small amplitude expansion, i.e., f' is not necessarily a "small" quantity.

A few properties, obtained by integrating by parts, will be very useful for further calculations:

$$\overline{f_x} = (\overline{f})_x = \frac{1}{L} \int_0^L f_x dx = f(L) - f(0) = 0\,; \tag{36}$$

$$\overline{f_x g} = \frac{1}{L} \int_0^L f_x g dx = -\overline{f g_x}\,; \qquad \overline{f_y g} = \overline{(fg)}_y - \overline{f g_y}\,; \tag{37}$$

$$\overline{AB} = \overline{A}\;\overline{B} + \overline{A'B'}\,. \tag{38}$$

Now we will look for equations for the averaged quantities like $\overline{\mathbf{u}}$ etc. by averaging the Oldroyd-B equations. The continuity equation gives

$$\overline{u_x} + \overline{v_y} = \overline{v_y} = 0\,, \Rightarrow \overline{v} = 0\,.$$

This is the first simplification obtained by introducing the concept of zonal averaging: the average y-velocity is zero and we need to consider only the equation for $\overline{u}$.

The averaged x-momentum equation is:

$$\overline{u_t} + \overline{uu_x} + \overline{vu_y} + \overline{p_x} = G(\overline{A}_{xx,x} + \overline{A}_{xy,y})\,. \tag{39}$$

The second, fourth, and fifth terms vanish while the third term reduces to average over product of disturbance parts:

$$\overline{uu_x} = \frac{1}{2}\left(\overline{u^2}\right)_x = 0\,,$$

$$\overline{p_x} = 0, \qquad \text{and} \qquad \overline{A}_{xx,x} = 0\,,$$

$$\overline{vu_y} = \overline{v}\;\overline{u}_y + \overline{v'u'_y} = 0 + (\overline{v'u'})_y - \overline{v'_y u'} = (\overline{v'u'})_y + \overline{u'_x u'} = (\overline{v'u'})_y + 0\,.$$

Thus we get the following x-momentum equation:

$$\overline{u}_t + (\overline{v'u'})_y = G\overline{A}_{xy,y}\,. \tag{40}$$

This equation shows another simplification of zonal averaging: the nonlinear terms contain only the disturbance parts and only one component of $\overline{\mathbf{A}}$ appears in the x-momentum equation.

The equations for $\overline{\mathbf{A}}$ can also be reduced to simpler form in similar fashion. Since only the component $\overline{A}_{xy}$ appears in the x-momentum equation, I will concentrate on the equation for $\overline{A}_{xy}$:

$$\overline{A}_{xy,t} + (\overline{v'A'_{xy}})_y = \overline{u}_y\overline{A}_{yy} + \overline{u'_y A'_{yy}} + \overline{v'_x A'_{xx}} \tag{41}$$

Again almost all the nonlinear terms, except the first one on right hand side, contain only the disturbance parts. In order to get a full set of equations, we will need equations for $\overline{A}_{yy}$ and $\overline{A}_{xx}$. But, at this stage I will introduce the small-amplitude expansion to study small-amplitude waves [which are $O(a)$] and their effect, accurate only up to $O(a^2)$, on the flow. We will see later that the above equations for $\overline{u}$ and $\overline{A}_{xy}$ form a closed set of equations after introducing the small-amplitude expansion.

For considering the small-amplitude waves, I will expand all the physical quantities in the following asymptotic expansion:

$$f = F + f_1 + f_2 + \cdots + f_n + O(a^{n+1})\,, \tag{42}$$

where F is the $O(1)$ background and $f_n = O(a^n)$. Each term of this expansion is decomposed into an average and a disturbance part:

$$f_n = \overline{f}_n + f'_n\,. \tag{43}$$

By definition, the background contains no disturbance part, i.e. $F' = 0$, while the first order quantities contain no mean part: $\overline{f}_1 = 0$. Thus, keeping only the terms up to $O(a^2)$ gives:

$$\overline{u} = U + \overline{u_2}\,;\; u' = u'_1 + u'_2\,;\; \overline{\mathbf{A}} = \mathbf{I} + \overline{A_2}\,;\; \mathbf{A}' = \mathbf{A}'_1 + \mathbf{A}'_2\,.$$

Thus we see that (40) and (41) correct up to $O(a^2)$ are:

$$\overline{u}_{2t} + G\overline{A}_{2xy,y} = (\overline{v'_1 u'_1})_y \,, \tag{44}$$
$$\overline{A}_{2xy,t} - \overline{u}_{2y} = -(\overline{v'_1 A'_{1xy}})_y + \overline{u'_{1y} A'_{1yy}} + \overline{v'_{1x} A'_{1xx}} \,. \tag{45}$$

We note that only the first order part of the disturbance and second order part of the mean appear in the above equations. This allows us to drop $(\ )_1$ from disturbances and $(\ )_2$ from means.

The above are wave equations for the means $\overline{u}$ and $\overline{A}_{xy}$. Note that there are source terms which appear as products of first order disturbances. Thus the effect on the mean-flow of the propagating vorticity waves and evanescent potential flow travels as a wave. The wave speed of this wave is $\sqrt{G}$ which is greater than $(l/\sqrt{k^2+l^2})\sqrt{G} = \sqrt{G}\sqrt{1-G/U^2}$ that is the y-component of group velocity of $O(a)$ vorticity waves. Thus we will distinguish between the waves of the mean part as "fast" vs. the waves of the disturbance part as "slow" waves.

The sources in the above equations appear only in terms of the first order disturbance parts. If we take a particular solution of the linearized equations (as given towards the end of Sec. 2), then we know the right hand sides of the above equations and we can solve them explicitly for $\overline{u}$ and $\overline{A}_{xy}$. But at this stage, I will introduce the ideas of Lagrangian mean averaging in contrast with the Eulerian zonal averaging $\overline{(\ )}$ that we have been using so far. The motivation for this step is that, in many cases, the equations in terms of Lagrangian averaged quantities are much simpler than those in terms of Eulerian means. We will soon see that such is indeed the case here.

4 Lagrangian mean averaging

In this section, I give a very brief introduction to Generalized Lagrangian Mean theory,[1, 2, 3] before applying it to the present problem. The GML theory obtains equations in terms of quantities averaged along the particle trajectory instead of averaging at a given spatial point, which is the case for Eulerian averaging. Thus, one of the main quantities to be used through-out the GLM theory is the disturbance related particle displacement field $\boldsymbol{\xi}(\boldsymbol{x},t)$. For example, the $\boldsymbol{\xi}'$ in (20) is the particle displacement field for the $O(a)$ vorticity waves.

The crux of the GLM theory is in the following two requirements:

- The field $\boldsymbol{\xi}(\boldsymbol{x},t)$ is defined in such a way that $\boldsymbol{x}+\boldsymbol{\xi}(\boldsymbol{x},t)$ is the actual position of the fluid particle whose mean position at time t is $\boldsymbol{x}$. Thus if we define $\boldsymbol{\Xi} = \boldsymbol{x}+\boldsymbol{\xi}(\boldsymbol{x},t)$, then we require that $\overline{\boldsymbol{\Xi}} = \boldsymbol{x}$. This is equivalent to requiring that $\boldsymbol{\xi}$ is a disturbance quantity.
$$\overline{\boldsymbol{\xi}}(\boldsymbol{x},t) = 0 \,. \tag{46}$$
- The other requirement is that $\boldsymbol{x}+\boldsymbol{\xi}(\boldsymbol{x},t)$ gives the actual trajectory of the material element of the fluid, i.e., the velocity of that point is the actual fluid velocity $\mathbf{u}^\xi$ at $\boldsymbol{x}+\boldsymbol{\xi}$.
$$\overline{D}^L \boldsymbol{\Xi} = \mathbf{u}^\xi \,, \tag{47}$$
where I have defined $\overline{D}^L := (\partial/\partial t + \overline{\mathbf{u}}^L \cdot \nabla)$.

Any field $f(\boldsymbol{x},t)$ is "lifted" to the actual particle position by defining

$$f^{\xi}(\boldsymbol{x},t) := f[\boldsymbol{x}+\boldsymbol{\xi}(\boldsymbol{x},t)]\,. \tag{48}$$

The *Lagrangian-mean* operator $\overline{(\ )}^{L}$ is then defined to be the average taken with respect the displaced position $\boldsymbol{x}+\boldsymbol{\xi}$, i.e.,

$$\overline{f(\boldsymbol{x},t)}^{L} := \overline{f^{\xi}(\boldsymbol{x},t)} = \overline{f[\boldsymbol{x}+\boldsymbol{\xi}(\boldsymbol{x},t)]}\,. \tag{49}$$

The main idea behind GLM theory is that the equations in terms of Lagrangian-averaged quantities should be of the same form as the original equations. As an example, consider a scalar quantity θ that is advected by the flow:

$$\frac{D\theta}{Dt} = \left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)\theta = 0\,. \tag{50}$$

With the above definitions and requirements, we can show that the Lagrangian-mean version of this equation is simply

$$\overline{D}^{L}\overline{\theta}^{L} = \left(\frac{\partial}{\partial t} + \overline{\mathbf{u}}^{L}\cdot\nabla\right)\overline{\theta}^{L} = 0\,. \tag{51}$$

This is much simpler than the Eulerian mean equation which contains products of disturbance parts from the nonlinear terms:

$$\left(\frac{\partial}{\partial t} + \overline{\mathbf{u}}\cdot\nabla\right)\overline{\theta} = -\overline{(\mathbf{u}'\cdot\nabla\theta')}\,. \tag{52}$$

For a vector or tensor field that is advected by the flow, the corresponding *mean* field can be defined in appropriate way so that the equation for advection remains form invariant under Lagrangian averaging.[3] In the absence of relaxation, the symmetric tensor $\mathbf{A}$ satisfies the equation

$$\frac{\partial\mathbf{A}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{A} - (\nabla\mathbf{u})^{T}\cdot\mathbf{A} - \mathbf{A}\cdot\nabla\mathbf{u} = 0\,. \tag{53}$$

If we define the mean stress tensor by

$$\hat{A}_{ij} := \overline{\left(\frac{A^{\xi}_{mn}K_{mi}K_{nj}}{J}\right)}\,, \tag{54}$$

where J is the Jacobian of the transformation $\boldsymbol{x}\to\Xi=\boldsymbol{x}+\boldsymbol{\xi}(\boldsymbol{x},t)$:

$$J = \left|\frac{\partial(\boldsymbol{\Xi})}{\partial(\boldsymbol{x})}\right| = \left|\delta_{ij}+\xi_{i,j}\right|\,, \tag{55}$$

and K_{mn} are the cofactors of the above matrix, then it can be verified that this mean vector is advected by the Lagrangian-mean velocity field, i.e.,

$$\frac{\partial\hat{\mathbf{A}}}{\partial t} + \overline{\mathbf{u}}^{L}\cdot\nabla\hat{\mathbf{A}} - \left(\nabla\overline{\mathbf{u}}^{L}\right)^{T}\cdot\hat{\mathbf{A}} - \hat{\mathbf{A}}\cdot\nabla\overline{\mathbf{u}}^{L} = 0\,. \tag{56}$$

Now we transform the wave equations for $\overline{u}$ and $\overline{A}_{xy}$ into equations for the Lagrangian-mean quantities $\overline{u}^L$ and $\hat{A}_{xy}$ defined as follows:

$$\overline{u}^L := \overline{u} + (\overline{\eta' u'})_y \,,$$
$$\hat{A}_{xy} := \overline{A}_{xy} + (\overline{\eta' A'_{xy}})_y + \overline{\xi'_x \xi'_y} + \overline{\eta'_x \eta'_y} \,.$$

These definitions agree with the above definitions (49) and (54) up to $O(a^2)$.

The averaged equations for these Lagrangian mean quantities take very simple form as seen by using the linear equations [(7)-(9) and $\nabla \cdot \boldsymbol{\xi} = 0$] for the $O(a)$ disturbance fields and the properties (36)-(38) of the Eulerian averaging:

$$\overline{u}^L_t - G\hat{A}_{xy,y} = (\overline{\eta'_x p'})_y =: S_y(y,t) \,, \tag{57}$$
$$\hat{A}_{xy,t} - \overline{u}^L_y = 0 \,. \tag{58}$$

For example,

$$\begin{aligned}\hat{A}_{xy,t} - \overline{u}^L_y &= -(\overline{v' A'_{xy}})_y + 2\overline{v'_x \xi'_x} + 2\overline{u'_y \eta'_y} \\ &\quad + (\overline{v' A'_{xy}})_y + (\overline{u'_y \eta'})_y + (\overline{v'_x \eta'})_y \\ &\quad + \overline{u'_x \xi'_y} + \overline{u'_y \xi'_x} + \overline{v'_x \eta'_y} \\ &\quad + \overline{v'_y \eta'_x} - (\overline{u'_y \eta'})_y - (\overline{u' \eta'_y})_y \\ &= 0 \,.\end{aligned}$$

A very similar, though tedious, calculation verifies the other equation. Using the previous linear solution, we can get the source inside the fully-developed wave-front to be:

$$S_y(y,t) = (\overline{\eta'_x p'})_y = h_0 k^2 G e^{-ky} \left[\sin(ly) - \cos(ly) \sqrt{\frac{U^2}{G} - 1} \right] \,. \tag{59}$$

Thus we see that the source term drops off exponentially because it gets contribution only from to the evanescent potential flow. Far enough from the boundary, these are source-free wave equations. This is in contrast to the much more difficult situation of the wave equations (44)-(45). The source terms for those equations get contributions from both the potential flow and the vorticity waves and are present even far from the boundary.

5 Spin-up problem

I will present the solution of the wave equations for the fast waves with the perturbation turned on at $t = 0$. The boundary condition of no tangential stress ($A'_{xy} = 0$ at the boundary) gives the boundary condition for $\hat{A}_{xy}$. I define the *drag* D to be $G\hat{A}_{xy}$ at the boundary:

$$D := G\hat{A}_{xy}|_{y=0} = 2G\overline{\xi'_x \eta'_x}\ \big|_{y=0} = \frac{2h_0^2 k^2 G^2}{U^2} \sqrt{\frac{U^2}{G} - 1} > 0$$

First I solve equations (57)-(58) dropping the source term S_y but with the boundary condition

$$\hat{A}_{xy}(y=0,t) = \frac{D}{G} H(t) \,, \tag{60}$$

where $H(t)$ is the Heaviside step function. This gives the following solution:

$$\hat{A}_{xy}(y,t) = \frac{D}{G} H(\sqrt{G}t - y)\,, \tag{61}$$

$$\overline{u}^L(y,t) = -\frac{D}{\sqrt{G}} H(\sqrt{G}t - y)\,. \tag{62}$$

Thus the total mean-flow (including the $O(1)$ and $O(a^2)$ parts) is

$$\overline{u}^L = \begin{cases} U - D/\sqrt{G} & \text{if } \; y < t\sqrt{G}\,, \\ U & \text{if } \; y > t\sqrt{G}\,. \end{cases} \tag{63}$$

Now solving these equations by keeping the source term $\partial S(y,t)/\partial y$ but with the boundary condition $\hat{A}_{xy}(y=0,t) = 0$, we get the following solution:

$$\hat{A}_{xy}(y,t) = \frac{1}{G}\left[S(y+\sqrt{G}t) + S(|y-\sqrt{G}t|)\right]\,, \tag{64}$$

$$\overline{u}^L(y,t) = \frac{1}{\sqrt{G}}\left[S(y+\sqrt{G}t) + S(|y-\sqrt{G}t|)\right]\,. \tag{65}$$

Since $S(y,t)$ decays exponentially in y, the effect of this term is seen only locally near the wave front. The main effect on the mean-flow is because of the drag at the boundary as given by (62).

Now, I plot the drag as a function of *Mach number* $M := U/\sqrt{G}$ in Fig. 1. We see that the drag reduces as a function of velocity for large background velocity U. Also for a fixed U, decreasing G leads to increasing Mach number and decreasing drag. This suggests that there might be an interesting connection of this problem to drag reduction.

Now we look at the energy conservation for the linear equations. Averaging (10) over x and integrating over y we get:

$$\begin{aligned}
&\frac{d}{dt}\int_0^{v_g t}\left(\frac{1}{2}\overline{\mathbf{u}'^2} + \frac{G}{4}\overline{\mathrm{tr}(\mathbf{A}'^2)}\right)dy + \int_0^{\infty}\left(\overline{v'p'} + G\overline{u'A'_{xy}} + G\overline{v'A'_{yy}}\right)_y dy \\
&= v_g\left(\overline{u'^2} + \overline{v'^2}\right) + \left(\overline{v'p'} - G\overline{u'A'_{xy}} - G\overline{v'A'_{yy}}\right)|_{y=0}^{y=\infty} \\
&= -(2h_0^2k^2G^2/U)\sqrt{U^2/G - 1} + UG\hat{A}_{xy}|_{y=0} \\
&= -UD + UG\hat{A}_{xy}|_{y=0} \\
&= 0\,.
\end{aligned}$$

This shows that the energy at $O(a^2)$ is carried by the $O(a)$ vorticity waves. This can be seen from, for example, the kinetic energy term $\mathbf{u}^2/2$. The contribution to this term from the $O(a^2)$ mean-flow $\overline{\mathbf{u}}$ is only $O(a^4)$. Thus the $O(a^2)$ contribution comes only from the $O(a)$ disturbance solution.

The momentum balance is given by integrating the momentum equation:

$$\begin{aligned}
&\frac{d}{dt}\int_0^{\sqrt{G}t}\overline{u}^L dy - \int_0^{\infty} G\hat{A}_{xy,y}dy - \int_0^{\infty} S_y dy \\
&= \sqrt{G}\overline{u}^L - G\hat{A}_{xy}|_{y=0}^{y=\infty} - S|_{y=0}^{y=\infty} \\
&= \sqrt{G}\overline{u}^L + G\hat{A}_{xy}|_{y=0} \\
&= 0\,.
\end{aligned}$$

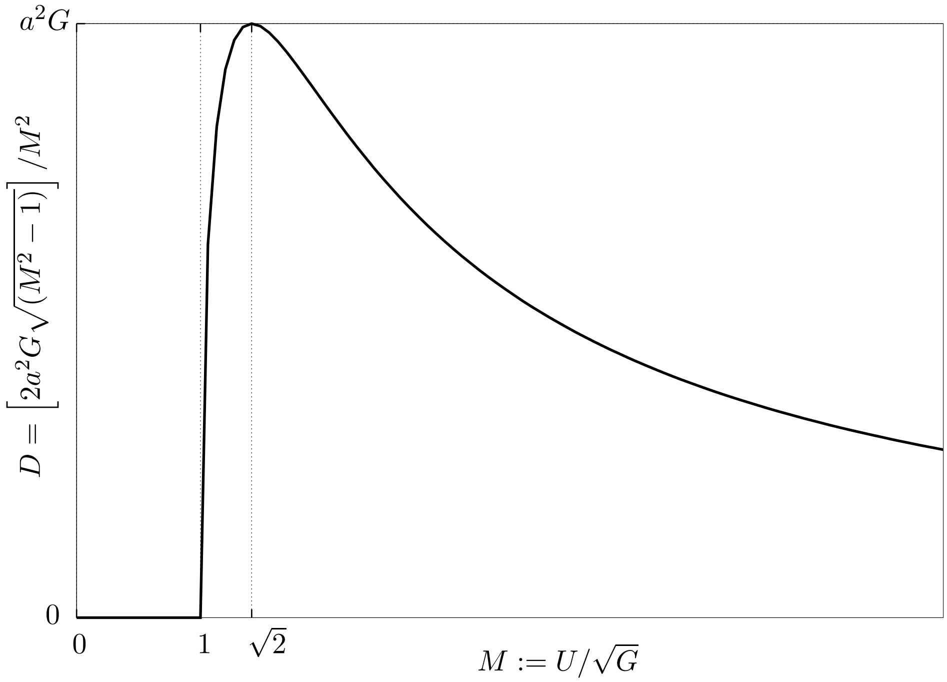

Figure 1: Drag on the mean-flow as a function of Mach number

Thus we see that the momentum at $O(a^2)$ is carried by the fast moving mean-flow response. This is because momentum is linear in, for example, $\mathbf{u}$ and the $O(a)$ disturbance part does not make any contribution.

The curious difference between the speeds of propagation of the $O(a)$ disturbance waves and $O(a^2)$ mean-flow response gives rise to separation between energy and momentum of the flow at $O(a^2)$. Suppose that the perturbation is kept on from $t = 0$ to $t = T$. Also assume T to be large enough for the stationary waves to develop fully. Then, at some later time $t \gg T$, the wavefront of the slow $O(a)$ vorticity waves will be traveling at a speed $v_s := \sqrt{G}\sqrt{1 - G/U^2}$ and these are the waves that carry the energy from the boundary. But the fast $O(a^2)$ waves (the mean-flow response), which carry the momentum from the boundary, will be traveling at a speed $\sqrt{G}$. Also, the effect on the mean-flow is seen even before the $O(a)$ waves arrive! This is shown in Fig. 2.

6 Conclusion

We have studied the various phenomena associated with waves propagating in the inviscid relaxation-less Oldroyd-B fluid. One of the main results is that the waves do not directly affect the mean-flow in the sense that the region where the mean-flow is affected can be separate from the region where the waves actually exist. The results were obtained by using the Generalized Lagrangian Mean theory.

There are several directions in which these results can be extended. Studying the full Oldroyd-B equations (with viscosity and relaxation) will be interesting. This might change

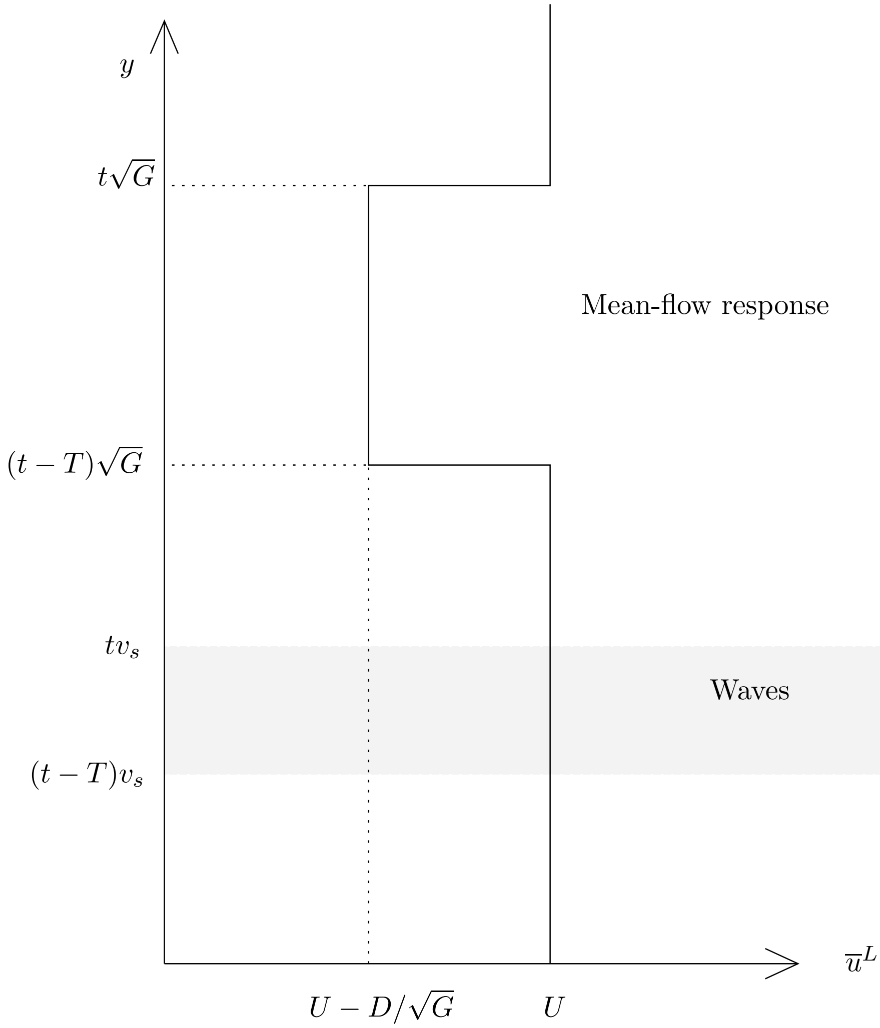

Figure 2: Mean-flow response due to the waves. Here v_s is the speed of the vorticity waves, shown by the shaded region.

the results significantly because we will need to use the no-slip boundary condition instead of free-slip condition. The inviscid relaxation-less model can be studied in the Hamiltonian formulation [using a non-canonical Poisson bracket and the Hamiltonian given by the left hand side of (4)]. Such an approach is developed in [4]. Studying the Lagrangian-mean theory in this Hamiltonian formulation can give insights into the (pseudo)energy and momentum equations. The interesting result about decrease in drag as a function of velocity can have some implications for turbulent drag reduction!

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