GFD 2012 Lecture 1: Dynamics of Coherent Structures and their Impact on Transport and Predictability

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1 Introduction

1.1 What is a Coherent Structure?

Structures are ubiquitous in planetary atmospheres, oceans, and stars (figure 1). These structures are of great interest because some of them have significant *direct* human impact (e.g. hurricanes, storms, Jet Stream). In addition, studying structures can provide insight into understanding and modeling other high-impact phenomena such as climate and weather. For example, sub-gridscale parametrization is an important part of ocean modeling; however, it has been observed that increasing resolution (and hence the accessible Reynolds number) drastically changes the flow field and results in an 'explosion' in the population of coherent vortices (figure 2).

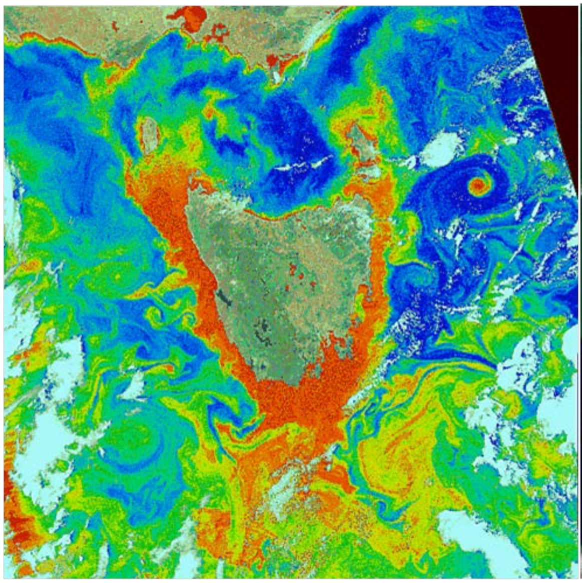
There is no rigorous definition of a coherent structure. In general, the best way to categorize them is based on the human brain, and employing the principle of 'you know one when you see one'. On the whole, structures *cannot* be derived from the underlying partial differential equations: some conclusions can be drawn from these governing equations, but in general we need experimental and numerical observations to guide a theoretical study of structures.

In general, simple systems provide a road map for more complex physical systems on a planetary scale, such as oceans and atmospheres. Therefore, the overall approach is to seek the generic properties of these planetary fluids, and study simplified systems. We then extrapolate these simple systems back to the more complex large-scale physical systems of interest.

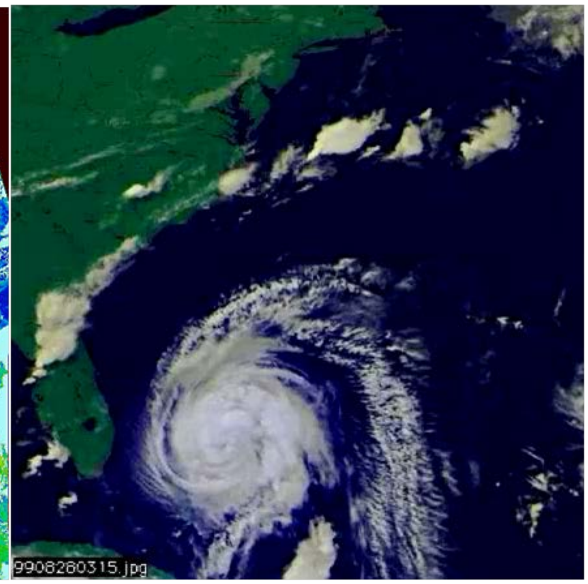
1.2 Properties of Coherent Structures

The term 'coherent structure' was first coined by [1] for vortices in a free shear layer. Below we give some properties of coherent structures, following [2]. It should be noted that the properties listed here are suggestions, and are probably both incomplete and overly exclusive, such that some things we would like to include as structures are either not included or are ruled out.

- Coherent structures are recurrent.

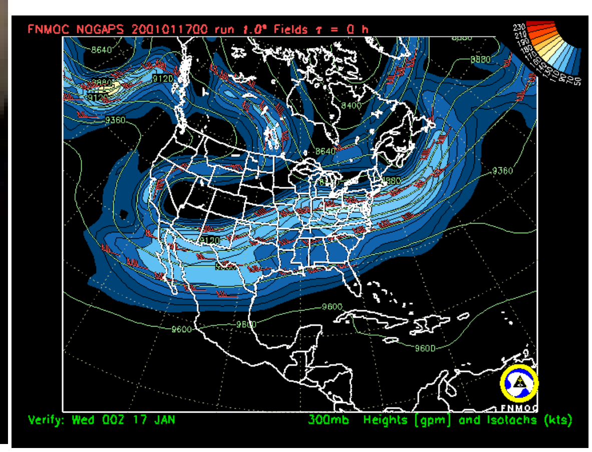


(a) An oceanic eddy off the coast of Tasmania



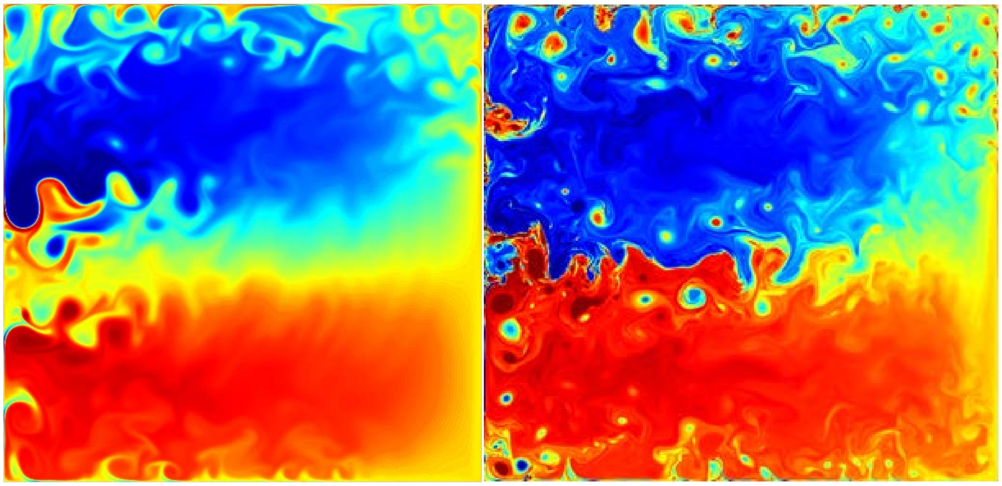
(b) Hurricane Dennis off the coast of Florida

(c) The Great Red Spot of Jupiter



(d) The Jet Stream over North America

Figure 1: Examples of structures in planetary atmospheres and oceans.



(a) 6.3 km resolution $\sim 1/12^o$ (b) 1.6 km resolution $\sim 1/48^o$

Figure 2: The effect of increasing numerical resolution in a simulation of ocean gyre, taken from [3].

- Coherent structures are spatially localized and isolated, as opposed to waves with a single Fourier mode„ which are not. Solitary waves, which are localized and contain many Fourier modes are an example of a coherent structure.
- Coherent structures are a preferred state of the nonlinear dynamics: they are either close to stationary flow configurations, or self-similarly evolving states, which are robust to perturbations.
- Coherent structures are dynamically self-organizing, and thus not characteristic of any forcing.
- Coherent structures are long-lived in a Lagrangian frame, meaning that the time scale over which the structure decays is much longer than the typical Eulerian time scales of the flow (e.g. the Eulerian rotation period of the vortex). Structures are therefore weakly dissipative on Eulerian timescales.

1.2.1 Coherent Structures and Turbulence Theory

In traditional theories of turbulence, flows are treated as random, and a random phase approximation in (Fourier) wavenumber space is typically used to analyze them. However, coherent structures, as defined above, are *local* in physical space, and therefore a random phase approximation will destroy the physical localization. For example, a step function in physical space has a wide spectrum in Fourier space (Heisenberg's uncertainty principle).

1.2.2 Recognition Algorithms

Structures can be identified by a number of different techniques. The most common approach is using subjective automated algorithms. These algorithms work by taking predefined criteria and thresholds for the various properties outlined above. Such a method has an inherent subjectivity. A good recognition algorithm will be robust to small changes in the criteria such as the specific values of the thresholds. Other methods include human identification by eye, Lagrangian coherent structure theory, wavelet theory, and various statistical procedures.

2 Two-Dimensional Fluid Dynamics

The large-scale dyynamics of planetary and oceanic flows are dominated by rotation and stable density stratification. Such flows are charactrized with velocity scale V, length scales L (horizontal) and H (vertical), Coriolis parameter $f = 2\Omega \sin\theta$ (where Ω is the frequency of rotation and θ is the latitude), and Brunt-Väisälä frequency N. The Brunt-Väisälä frequency describes the frequency of oscillation of a displaced parcel of fluid in a stable stratified density field $\rho(z)$, and is given by $N^2 = (-g/\rho)\,\partial\rho/\partial z$, where g is the gravitational acceleration.

The effects of rotation are described by the Rossby number Ro, which can be thought of as a ratio of the timescales for rotation and advection, and is given by

$$Ro = \frac{U}{Lf}. \tag{1}$$

If $Ro \ll 1$, then the rotation timescale $1/f$ is much shorter than the advection timescale L/U, and the effects of rotation dominate.

The effects of stable stratification are described by the Froude number F, which can be thought of a ratio of the timescales for oscillation in the background stratification and advection, and is given by

$$F = \frac{U}{HN}. \tag{2}$$

If $F \ll 1$, then the stratification in strong.

In the limits $Ro \ll 1$ and $F \ll 1$, the system is marked by rapid rotation and strong stable stratification, which results in significant (spatial) anisotropy in the flow. In particular, the vertical velocity w is smaller than the horizontal ones (i.e. $w \ll u, v$). If the extreme limit of $w = 0$, we can consider the system as two-dimensional in the $x - y$ plane.

2.1 2D Fluid Equations

We assume that the flow $\boldsymbol{u}(x, y, t) = (u, v)$ is incompressible

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0, \tag{3}$$

and that the density ρ is constant. Without loss of generality, we set $\rho = 1$. The flow satisfies the Navier-Stokes momentum equation,

$$\frac{D\boldsymbol{u}}{Dt} = \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\,\boldsymbol{u} = -\nabla p + \nu \nabla^2 \boldsymbol{u}, \tag{4}$$

where p is the pressure, ν is the viscosity, and rotation is ignored for now (i.e. $f = 0$). We define the vorticity $\boldsymbol{\omega}$ to be

$$\boldsymbol{\omega} = \nabla \wedge \boldsymbol{u} = \omega \hat{\boldsymbol{z}}, \tag{5}$$

Due to incompressibility (3) the flow can be described by a streamfunction $\psi(x, y, t)$ as $(u, v) = (-\partial\psi/\partial y, \partial\psi/\partial x)$. With respect to this streamfunction, the vorticity is given by

$$\omega = \nabla^2 \psi. \tag{6}$$

Taking the curl of (4) gives the vorticity equation,

$$\frac{D\omega}{Dt} = \frac{\partial \omega}{\partial t} + J[\psi, \omega] = \nu \nabla^2 \omega, \tag{7}$$

where the Jacobean J is given by

$$J[\psi, \omega] = \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x}. \tag{8}$$

In the inviscid limit, $\nu \to 0$, (7) reduces to

$$\frac{D\omega}{Dt} = 0, \tag{9}$$

and vorticity is conserved following the flow.

In general, the flow is described by the vorticity equation (7) and the equation relating vorticity to the streamfunction (6). Often, in more situations beyond pure 2d flows, conservation equations for 'potential vorticity' can be derived that are analogous to (7), but the relationship between potential vorticity and velocity will be different to (6) (e.g. see 2.1.1).

Finally, we define the circulation Γ_C around a closed curve C to be

$$\Gamma_C = \oint_C \boldsymbol{u} \cdot \boldsymbol{dl} = \int_S \boldsymbol{\omega} \cdot \boldsymbol{dS}, \tag{10}$$

where the closed curve C, with line element $\boldsymbol{dl}$, contains an area S, with area element $\boldsymbol{dS}$.

2.1.1 Effect of Rotation

In 2D fluid dynamics, if the Coriolis parameter f is a constant (the 'f–plane approximation'), then it can be absorbed into a modified pressure p in (4), becuase the Coriolis term $f(\boldsymbol{u}\wedge\hat{\mathbf{z}})$ can be written as a perfect gradient $f\nabla\psi$. In this case, the flow is still described by (6) and (7). However, if f varies by latitude, e.g. as $f(y) = f(y_o) + \beta y$ where $\beta = (\partial f/\partial y)_{y_o}$ (β–plane approximation), then (7) will be replaced with an equation conserving *potential* vorticity $q = \omega + \beta y$ (i.e. $Dq/Dt = 0$).

2.2 Steady Inviscid Solutions

On an f-plane, if the flow is steady and inviscid, then (7) reduces to

$$J[\psi, \omega] = 0. \tag{11}$$

Therefore, any parallel flow, in which the streamfunction is a function of x or y only, is a solution of the equations. For example, zonal or meridional jets are solutions.

Note, however, that under a β–plane approximation (described above), meridional jets ($\psi = \psi(x)$) are no longer solutions of the equations. To the degree that coherent structures are steady-solutions of the inviscid equations, this explains why coherent jets tend to zonal rather than meridional.

2.3 Vortices

In polar coordinates (r, θ), the steady inviscid governing equation (11) and the vorticity (6) can be written as

$$J[\psi, \omega] = \frac{1}{r}\left(\frac{\partial \psi}{\partial r}\frac{\partial \omega}{\partial \theta} - \frac{\partial \psi}{\partial \theta}\frac{\partial \omega}{\partial r}\right) = 0, \tag{12}$$

$$\omega = \nabla^2 \psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2}. \tag{13}$$

Therefore, any axisymmetric flow $\psi = \psi(r)$ satisfies the steady inviscid equations (12) and (13). Again, to the degree that coherent structures are steady-solutions of the inviscid equations, we expect to see axisymmetric vortices.

2.3.1 Gaussian vortex

Consider the axisymmetric vortex given by

$$\omega(r) = \frac{\Gamma}{2\pi r_0^2}\mathrm{e}^{-r^2/2r_0^2}, \tag{14}$$

which is described by two constant parameters: the circulation Γ; and the size r_0. The velocity field is purely azimuthal $\boldsymbol{u} = u_\theta \boldsymbol{e}_\theta$, and can be found from inverting the curl (13) (analogous to the Biot-Savart law in electromagnetism). This operation gives a constant of integration, which is chosen to ensure that the velocity is bounded at the origin. The velocity is therefore given by

$$u_\theta(r) = \frac{\Gamma}{2\pi r}\left[1 - \mathrm{e}^{-r^2/2r_0^2}\right]. \tag{15}$$

Note that, with vortices, we often define a vorticity ω, and infer the velocity u_θ. Equation (15) shows that the velocity is zero at the origin and increases initially linearly with r. The velocity is maximum at r_0, and then decays like $1/r$.

2.3.2 Point Vortex

In the limit $r_0 \to 0$, we can consider the Gaussian vortex to be a point vortex, with vorticity ω given by

$$\omega(\boldsymbol{x}) = \Gamma \delta(\boldsymbol{x}). \tag{16}$$

The corresponding velocity is given by

$$u_\theta(r) = \begin{cases} \Gamma/2\pi r & r \neq 0 \\ 0 & r = 0 \end{cases} \tag{17}$$

Suppose we have N point vortices, each with circulation Γ_i and location $\boldsymbol{x}_i$. Then the vorticity at position $\boldsymbol{x}$ is obtained from the superposition of ω_i

$$\omega(\boldsymbol{x}) = \sum_{i}^{N} \Gamma_i \delta(\boldsymbol{x} - \boldsymbol{x}_i), \tag{18}$$

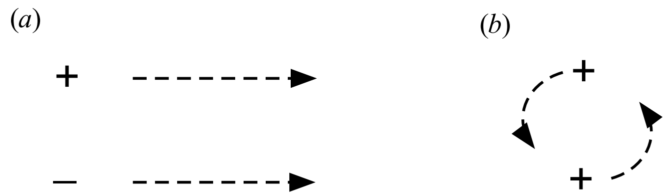


Figure 3: Schematic showing point vortex pairs: (a) two vortices of equal and opposite circulation, translate without changing their separation; (b) two vortices of equal circulation, rotate without changing their separation.

and the corresponding velocity field outside all point vortice is given by

$$\boldsymbol{u}(x, y) = \sum_{i}^{N} \frac{\Gamma_i}{2\pi} \left[\frac{-(y-y_i)\,\hat{\boldsymbol{x}} + (x-x_i)\,\hat{\boldsymbol{y}}}{(x-x_i)^2 + (y-y_i)^2} \right], \quad (x, y) \neq (x_i, y_i) \tag{19}$$

and the velocity at point vortex j is given by

$$\boldsymbol{u}(x_j, y_j) = \sum_{i \neq j}^{N} \frac{\Gamma_i}{2\pi} \left[\frac{-(y-y_i)\,\hat{\boldsymbol{x}} + (x-x_i)\,\hat{\boldsymbol{y}}}{(x-x_i)^2 + (y-y_i)^2} \right]. \tag{20}$$

Consider now the inviscid time-dependent governing equation,

$$\frac{\partial \omega}{\partial t} + J\left[\psi, \omega\right] = 0. \tag{21}$$

Inserting (18) into (21), and balancing terms, gives

$$\frac{\partial \Gamma_i}{\partial t} = 0, \qquad \text{and} \qquad \frac{D\boldsymbol{x}_i}{Dt} = \boldsymbol{u}(\boldsymbol{x}_i), \tag{22}$$

which shows that the circulation of each vortex remains constant, and each vortex moves with the velocity that is induced from the other vortices at that point.

Examples of the motion of two point vortices are given in figure 3. A pair of point vortices with equal but opposite circulation separated by a distance d translates at the speed of $\Gamma/2\pi d$ without changing the separation. The direction of translation can be easily inferred by finding the direction of velocity induced by one vortex on the other one. On the other hand, a pair of vortices with equal circulation of the same sign rotates around their centre of vorticity with a period of $2\pi^2 d^2/\Gamma$.

References

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