Lecture 4: Radiative-Convective Equilibrium and Tropopause Height

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We now consider what the effect of convection might be on all the concepts and solutions found in lecture 2. Because our interest in mainly in the large scale structure of the atmosphere we will take a somewhat simplistic view of convection and suppose that it acts to restore an unstable lapse to something that is neutrally stable, that lapse rate being given by either the dry adiabatic or moist adiabatic lapse rate. Readers interested in finding out more about convection and radiative-convective equilibrium should consult Kerry Emanuel's lecture notes.

1 Radiative-convective equilibrium

In lecture 2 we found that in radiative equilibrium the temperature falls off very rapidly with height in the lower atmosphere, so much so that it is likely to be convectively unstable. We imagine the atmosphere will convect and that the lapse rate will adjust until it is stable, as in Fig. 1, up to some height H_T. Sometimes, either instead of or in addition to, heat may be transported upwards by large-scale motion such as baroclinic waves. In either case, let us suppose that the dynamics acts such as to produced constant lapse rate up to some height H_T, which we will later associate with the tropopause. We wish to obtain an expression for that height. That is, we seek a solution for which

$$z \le H_T : \qquad T = T_s - \Gamma z \tag{1a}$$

$$z > H_T : \qquad \text{Radiative equilibrium, satisfying (lec.2:26) and (lec.2:32)} \tag{1b}$$

Further, since we are imposing a convective heat flux, we can suppose that at the surface the temperature is continuous, so that the ground temperature is such that $\sigma T_g^4 = U(z = 0)$.

To obtain a solution we might just think of adjusting the lapse rate in (lec2:fig.7) so that there is no net heating, and this may indeed be what convection does on a short timescale. However, an overall radiative balance is not necessarily then achieved, so that the system will then evolve further. The variable in this equation is H_T, and this can be adjusted until (lec.2:36) is satisfied, with the outgoing radiation The solution of these equations requires an iterative approach and the algorithm is as follows.

1. First solve the radiative transfer equations for radiative equilibrium.
2. Make a guess for the height of the tropopause, and hence obtain the temperature all the way down to the ground.

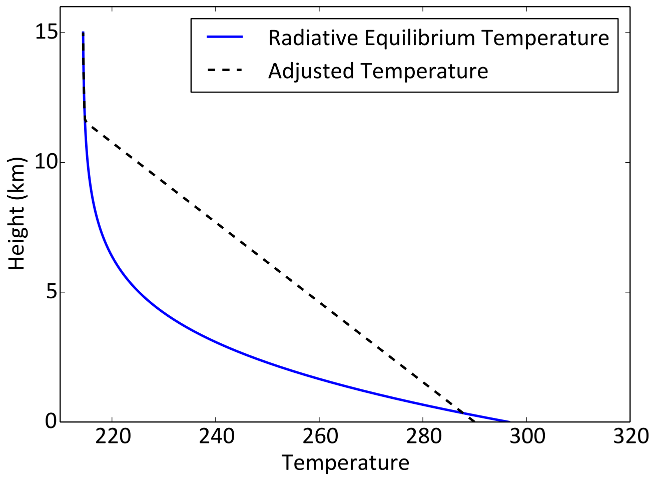

Figure 1: Radiative equilibrium temperature (solid curve) calculated using (lec.2:36), with an optical depth of $\tau_0 = 8/3$, $H_a = 2\,\mathrm{km}$ and a net incoming solar radiation of $239\,\mathrm{W\ m}^{-2}$. The dashed line shows a schematic adjusted temperature with a lapse rate of $6.5\,\mathrm{K\ km}^{-1}$ up to a tropopause (at about 11 km here) and a radiative equilibrium temperature in the stratosphere.

3. Integrate the radiative transfer equations down from the top. The outgoing radiative balance is achieved this way, but there is no balance at the surface if temperature is continuous. That is, $\sigma T_g^4 \neq U(z = 0)$.

4. Change the height of the tropopause, find another solution, and iterate until the surface radiative balance is achieved.

An alternative is to specify the surface temperature and integrate the radiative transfer equations up along a given lapse rate from the bottom to a certain height, beyond which we suppose that radiative equilibrium holds. This procedure will not give the correct outgoing radiation, so the procedure must again be iterated.

1.1 Global Warming

Without actually solving the RCE equations we can make an important deduction as to what happens to the height of the tropopause under global warming, that is what happens when additional carbon dioxide is added to the atmosphere. If the atmosphere stays in radiative balance (which it will in the long term) then the outgoing radiation remains the same. If the stratosphere has a small optical depth then its temperature stays the same from (lec.2:40). Therefore *the temperature of the tropopause must stay the same!* However, the height of the emitting temperature must increase, because the emissivity of the lower atmosphere increases, and the photons that reach space come, on average, from a higher level in the troposphere. And, as a consequence, the troposphere warms as illustrated in Fig. 2. But if the temperature of the tropopause is to stay the same then its height must increase, and a simple calculation tells us by how much.

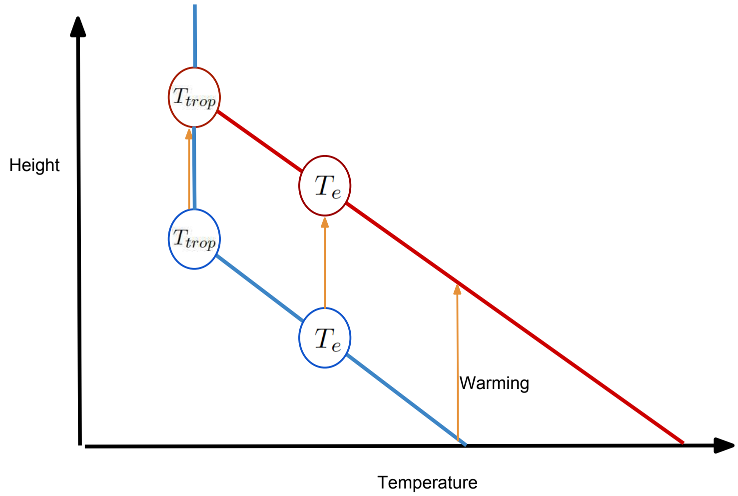

Figure 2: Schematic of temperatures before (blue line) and after an increase in optical depth of the atmosphere, such as happens in global warming. The troposphere warms but the emitting temperature stays the same. Hence the tropopause temperature stays the same and the height of the tropopause increases.

If the lapse rate stays the same then the tropopause height will increase by an amount ΔH_T given by

$$\Delta H_t = \frac{\Delta T_s}{\Gamma} \tag{2}$$

where ΔT_s is the change in surface temperature. If we allow the lapse rate to change also, then

$$\Delta H_t = \frac{\Delta T_s}{\Gamma} - H_t \frac{\Delta \Gamma}{\Gamma} \tag{3}$$

or

$$\frac{\partial H_t}{\partial T_s} = \frac{1}{\Gamma} - \frac{H_T}{\Gamma}\frac{\partial \Gamma}{\partial T_s}. \tag{4}$$

If we suppose that Γ is the moist adiabatic lapse rate then we can calculate this expression analytically, and some results are shown in Fig. 3, where the lapse rate is assumed constant with height and a function of surface temperature. It is interesting that the increase in tropopause height is quite significant – about 400 m per degree – and that both the direct temperature effect and the lapse rate effect are important (at least in regions where the lapse rate is moist adiabatic). An increase in tropospheric height is one of the most robust results we have concerning changes of the structure of the atmosphere under global warming, as discussed more in Vallis *et al.* (2014).

2 The Height of the Tropopause

We now provide an approximate, analytic, expression for the height of the tropopause.[1] We assume the following.

[1] This section is joint work with Pablo Zurita-Gotor.

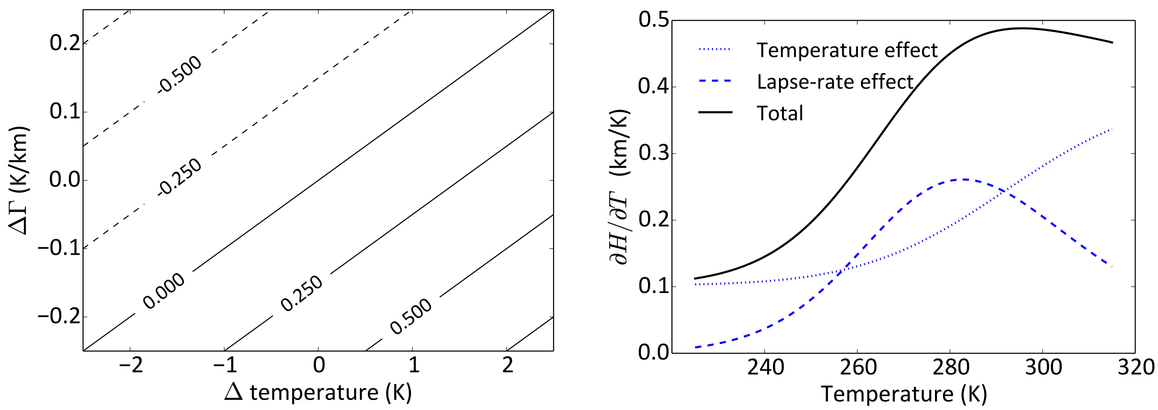

Figure 3: (a) Contours of change in tropopause height (km) as a function of temperature change and lapse rate change, calculated using (3). (b) Rate of change of tropopause height with temperature ($\partial H_t/\partial T$) as a function of temperature, calculated using (4).

1. Single column (so a one-dimensional calculation).
2. Grey atmosphere with an optical thickness that decays exponentially with height.
3. A specified lapse rate to some height H_T, beyond which there is radiative equilibrium.
4. An optically thin atmosphere in the upper troposphere and stratosphere.
5. An overall radiative balance. So the outgoing IR radiation is specified (equal to net incoming solar).
6. No surface temperature discontinuity. So ground temperature equals surface air temperature ($T_g = T_s$), and the upwards radiation at $z = 0$ is given by σT_g^4.

2.1 Algorithm

To find an exact solution the equations must be iterated, and an algorithm for that is as follows.

1. First numerically integrate (lec.2:33) to obtain a radiative equilibrium solution.
2. Guess a height for the tropopause and thus obtain a temperature at all levels below that, including the ground, using the given lapse rate.
3. Calculate the radiative fluxes by integration of (lectire 2:33) down from the top. The upwards radiation at the ground will in general not equal σT_g^4.
4. Adjust the height of the tropopause and repeat step (2) and (3).
5. Iterate the calculation until a surface balance is achieved.

An alternative procedure is to guess a surface temperature and integrate the equations up, assuming a constant lapse rate up to a height H_T, with radiative equilibrium beyond. When this is done the temperature at H_T will not be the correct one, and outgoing radiation will not equal to the incoming radiation, and again we have to iterate.

2.2 Analytic approximation

The analytic approach involves obtaining an analytic expression for the outgoing radiation for a given temperature profile along the lines of (lec.2:30). The OLR so obtained will be a function of the height of the tropopause, and by making the expression equal to the incoming solar radiation we obtain an expression for the tropopause height. Instead of actually using (lec.2:30) it is easier to solve the equations approximately *ab initio.* We make one other approximation, that the value of B/U varies linearly from the tropopause (where its value is 0.5) to its value at the surface (where $B/U = 1$). Thus,

$$\frac{B}{U} = 1 - \frac{z}{2H_T}. \tag{5}$$

Numerical calculations suggest this is a decent approximation (can it be improved upon?). Rewrite (lec.2:26a) as

$$\frac{\mathrm{d}\log U}{\mathrm{d}\tau} = 1 - \frac{B}{U} = \frac{z}{2H_T}. \tag{6}$$

Using $\tau(z) = \tau_s \exp(-z/H_a)$ we obtain

$$\frac{\mathrm{d}\log U}{\mathrm{d}z} = -\frac{z}{2H_T H_a}\tau_s \exp(-z/H_a). \tag{7}$$

We can integrate this expression by parts to obtain a value of the upwelling radiation at the tropopause $U(H_T)$, namely

$$\log\left(\frac{U(H_T)}{U(0)}\right) = -\frac{\tau_s}{2H_T}\int_0^{H_T} \exp(-z/H_a)\,\mathrm{d}z \approx -\frac{\tau_s H_a}{2H_T}. \tag{8}$$

for $H_T \gg H_a$. This is an expression for the outgoing longwave radiation, and we see that the only variable in the equation is H_T – note that the upwelling radiation at the surface is given by the surface temperature, which is a function of the tropopause temperature, H_T and the lapse rate, Γ.

To obtain a closed form for the tropopause height assume that the stratosphere is optically thin and note that $U(H_T) = U(H_T) = 2\sigma T_T^4$ and $U(0) = \sigma T_g^4 = \sigma T_s^4$. Furthermore, T_T and T_s are related by $T_T = T_s - \Gamma H_T$. The left-hand side of (8) then becomes

$$\begin{aligned}\log\left(\frac{2\sigma T_T^4}{\sigma T_s^4}\right) &= \log 2 + 4\log\frac{T_T}{T_s} = \log 2 + 4\log\left(\frac{T_T}{T_T + \Gamma H_T}\right) \\ &\approx \log 2 - \frac{4\Gamma H_T}{T_T}.\end{aligned} \tag{9}$$

Using (9), (8) becomes

$$\log 2 - \frac{4\Gamma H_T}{T_s} = -\frac{\tau_s H_a}{2H_T} \tag{10}$$

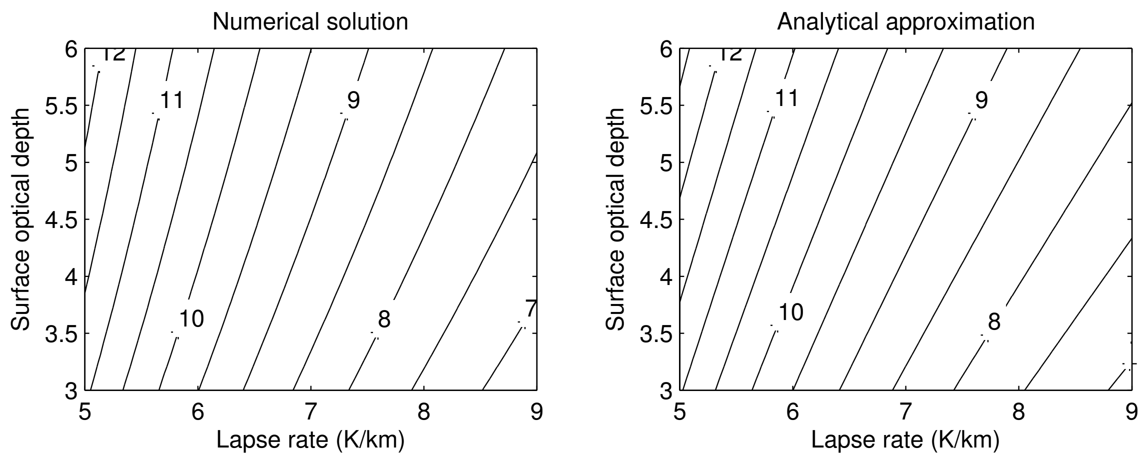

Figure 4: Analytic approximation and numerical calculation for tropopause height.

or

$$8\Gamma H_T^2 - \mathrm{C}\, H_T T_T - \tau_s H_a T_T = 0. \tag{11}$$

where $\mathrm{C} = 2\log 2 \approx 1.38$. The solution of this equation is

$$H_T = \frac{1}{16\Gamma}\left(\mathrm{C}T_T + \sqrt{\mathrm{C}^2 T_T^2 + 32\Gamma\tau_s H_a T_T}\right). \tag{12}$$

For Earth's atmosphere, $H_a \approx 2\,\mathrm{km}$, $\tau_s \approx 8/3$ and $\Gamma \approx 6.5\,\mathrm{K\ \ km}^{-1}$. All three terms in the quadratic are then approximately the same size and $H_T = 10.3\,\mathrm{km}$, which is in fact reasonably close to the exact numerical solution (obtained iteratively) of the radiative-convective equations (Fig. 4).

The numerical approximation of the logarithm in (9) can be improved by using T_m instead of T_T, where T_m is the temperature half way between the surface and the tropopause. However, we still want to have T_T as a parameter in the quadratic for H_T (because T_T is given if the OLR is known). Thus, we have to do some more algebraic fiddling and the upshot is that we get a quadratic similar to (11) but with different coefficients. [Student exercise. See also Vallis et al (2014) for another way to proceed.]

Once we have the tropopause height we can obtain an expression for the temperature everywhere in the troposphere, and the surface. We could then perform a calculation similar to that of section (lec.2:3.2) and obtain an analytic expression for how the surface temperature increases with carbon dioxide content, and the conditions for a runaway greenhouse effect. [With extensions this could be a student project.]

Optically thick and thin limits

The above approach allows us to be precise about what it means for an atmosphere to be optically thin or thick. Using (12) and approximating $C^2 = 2$ we easily find that the optically thick limit arises when

$$\tau_s H_a \gg \frac{T_T}{16\Gamma} \quad \text{whence} \quad H_T \approx \sqrt{\frac{T_T \tau_s H_a}{8\Gamma}} \tag{13}$$

The optically thin case has

$$\tau_s H_a \ll \frac{T_T}{16\Gamma} \quad \text{whence} \quad H_T \approx \frac{1.38 T_T}{8\Gamma}. \tag{14}$$

With parameters appropriate for Earth's atmosphere both of the above limits give estimates in the range 5–10 km , and note that they are additive effects. What is the interpretation of these expressions? Do they work on other planets? What is the role of lateral heat transport?

A number of these issues have been taken up by Shineng Hu in his summer project, and the interested reader is referred to his report for more details.

3 Lateral Transport

The actual tropopause height is determined by a combination of lateral heat transport and the RCE state above. Suppose we think of the energy balance of a column of air. If there is no horizontal divergence of heat flux into the column then we have the same situation as before, and the tropopause height is determined by the RCE argument. This in fact will be the situation somewhere in mid-latitudes where the horizontal heat flux is a maximum. At this particular location we have the RCE problem above.

Elsewhere there is a flux of heat into or out of the column and this will affect the tropopause height. As far as the column is concerned these effects are similar to changes in the outgoing longwave radiation, and so it would change the tropopause height in the same way that changing the outgoing longwave radiation would. That is, suppose you have separately solved the problem on how heat is transferred horizontally as well as the RCE problem above. In that case you know the outgoing radiation at any particular latitude and you know the lapse rate and so you can determine the tropopause height. In fact, this effect probably will not make a big difference to the tropopause height because the sensitivity of tropopause height on tropopause temperature is fairly weak. Thus, if the lapse rate is fixed independently of the horizontal dynamics the height of the tropopause will only be affected by these dynamics to a limited degree.

However, there are other ways that the dynamics affects things, and one is in the determination of the lapse rate itself. This may occur in either the tropics or the extratropics, where the mechanisms will be slightly different. In the tropics the prime determinant of the lapse rate is moist convection, and a simple possibility is that the lapse rate is given by the moist adiabatic lapse rate. However, this is itself a fairly strong function of temperature, so that the tropopause height becomes a function of temperature mainly through the effect that horizontal transfer has on the moist adiabatic lapse rate, not temperature itself.

Another argument (Held?) is that the moist static energy at the tropopause is almost the same as what it is at the surface, whence

$$N^2 \approx \frac{L q_s g}{c_p T_s H_T} \tag{15}$$

where H_T is, again, tropopause height and q_s and T_s are the surface values of water vapour and temperature. Note that H_T decreases with increasing stability in this expression, whereas it increases with stability (i.e., decreases with lapse rate) in the RCE expression (12). Only one value of the tropopause height is consistent with both.

In midlatitudes the situation is complicated because the heat transport is effected by baroclinic instability and this has a characteristic height. One possibility is to try to adjust things so that height of the tropopause is consistent both with baroclinic instability and with the radiative constraint. There are a number of possibilities that we might wish to consider (good student projects!).

Suppose that baroclinic instability is like the Eady problem. In this case the instability goes from the surface to the tropopause; there is no additional height scale.

But suppose that we have a β-plane. In this case the vertical scale is given by the 'Charney height' which, in a Boussinesq system, is

$$h = \frac{f^2\Lambda}{\beta N^2} \tag{16}$$

where Λ is the vertical shear of the zonal wind, $\Lambda = \partial U/\partial z$. We might argue that the height, h, in (16) must be the same as that given by the radiative constraint, and this gives us a theory of the stratification of the atmosphere. Just as in the tropics, h decreases with increasing stability so there is only one solution. If we suppose that $h = H_T$, where H_T is the height of the tropopause given by the radiative constraint, then using the thermal wind equation, $f\Lambda = -\partial b/\partial y$ and with $N^2 = \partial b/\partial z$ we find

$$H_T = -\frac{f\partial b/\partial y}{\beta\partial b/\partial z} \tag{17}$$

or

$$s = \frac{f}{\beta H_T} \sim \frac{a}{H_T}, \tag{18}$$

where s is the slope of the isopycnals and a is the Earth's radius. This means that an isopycnal roughly goes from the surface at the equatorward edge of the midlatitudes to the tropopause at the pole. This hypothesis and its friends have generated quite a lot of controversy in the community....

4 Appendix: Dry and Wet Lapse Rates

4.1 A dry ideal gas

The negative of the rate of change of the temperature in the vertical is known as the *temperature lapse rate*, or often just the lapse rate, and the lapse rate corresponding to $\partial\theta/\partial z = 0$ is called the *dry adiabatic lapse rate* and denoted Γ_d. Using $\theta = T(p_0/p)^{R/c_p}$ and $\partial p/\partial z = -\rho g$ we find that the lapse rate and the potential temperature lapse rate are related by

$$\frac{\partial T}{\partial z} = \frac{T}{\theta}\frac{\partial\theta}{\partial z} - \frac{g}{c_p}, \tag{19}$$

so that the dry adiabatic lapse rate is given by

$$\Gamma_d = \frac{g}{c_p}. \tag{20}$$

The conditions for static stability are thus:

$$\begin{array}{lll} \text{stability}: & \dfrac{\partial \widetilde{\theta}}{\partial z} > 0; \qquad \text{or} & -\dfrac{\partial \widetilde{T}}{\partial z} < \Gamma_d \\ \text{instability}: & \dfrac{\partial \widetilde{\theta}}{\partial z} < 0; \qquad \text{or} & -\dfrac{\partial \widetilde{T}}{\partial z} > \Gamma_d \end{array}, \tag{21a,b}$$

where a tilde indicates that the values are those of the environment. The atmosphere is, in fact, generally stable by this criterion: the observed lapse rate, corresponding to an observed buoyancy frequency of about $10^{-2}\,\mathrm{s}^{-1}$, is often about $7\,\mathrm{K\ km}^{-1}$, whereas a dry adiabatic lapse rate is about $10\,\mathrm{K\ km}^{-1}$. Why the discrepancy? One reason, particularly important in the tropics, is that the atmosphere contains water vapour.

4.2 Saturated lapse rate

The amount of water vapour that can be contained in a given volume is an increasing function of temperature (with the presence or otherwise of dry air in that volume being largely irrelevant). Thus, if a parcel of water vapour is cooled, it will eventually become saturated and water vapour will condense into liquid water. A measure of the amount of water vapour in a unit volume is its partial pressure, and the partial pressure of water vapour at saturation, e_s, is given by the Clausius–Clapeyron equation,

$$\frac{\mathrm{d}e_s}{\mathrm{d}T} = \frac{L_c e_s}{R_v T^2}, \tag{22}$$

where L_c is the latent heat of condensation or vapourization (per unit mass) and R_v is the gas constant for water vapour. If a parcel rises adiabatically it will cool, and at some height (known as the 'lifting condensation level', a function of its initial temperature and humidity only) the parcel will become saturated and any further ascent will cause the water vapour to condense. The ensuing condensational heating causes the temperature and buoyancy of the parcel to increase; the parcel thus rises further, causing more water vapour to condense, and so on, and the consequence of this is that an environmental profile that is stable if the air is dry may be unstable if saturated. Let us now derive an expression for the lapse rate of a saturated parcel that is ascending adiabatically apart from the effects of condensation.

Let w denote the mass of water vapour per unit mass of dry air, the mixing ratio, and let w_s be the saturation mixing ratio. ($w_s = \alpha e_s/(p - e_s) \approx \alpha_w e_s/p$ where $\alpha_w = 0.622$, the ratio of the mass of a water molecule to one of dry air.) The diabatic heating associated with condensation is then given by

$$Q_{cond} = -L_c \frac{\mathrm{D}w_s}{\mathrm{D}t}, \tag{23}$$

so that the thermodynamic equation is

$$c_p \frac{\mathrm{D}\ln\theta}{\mathrm{D}t} = -\frac{L_c}{T}\frac{\mathrm{D}w_s}{\mathrm{D}t}, \tag{24}$$

or, in terms of p and and T

$$c_p \frac{\mathrm{D}\ln T}{\mathrm{D}t} - R\frac{\mathrm{D}\ln P}{\mathrm{D}t} = -\frac{L_c}{T}\frac{\mathrm{D}w_s}{\mathrm{D}t}. \tag{25}$$

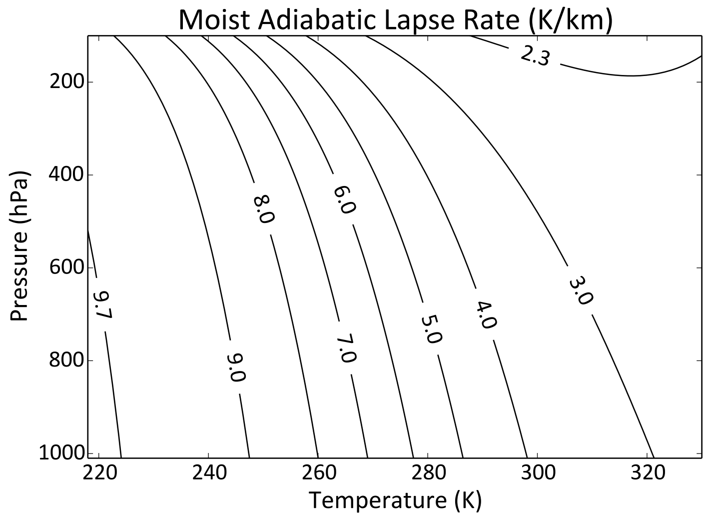

Figure 5: The saturated adiabatic lapse rate as a function of temperature and pressure when water (H_2O) is the condensate.

If these material derivatives are due to the parcel ascent then

$$\frac{\mathrm{d}\ln T}{\mathrm{d}z} - \frac{R}{c_p}\frac{\mathrm{d}\ln p}{\mathrm{d}z} = -\frac{L_c}{Tc_p}\frac{\mathrm{d}w_s}{\mathrm{d}z}, \tag{26}$$

and using the hydrostatic relationship and the fact that w_s is a function of T and p we obtain

$$\frac{\mathrm{d}T}{\mathrm{d}z} + \frac{g}{c_p} = -\frac{L_c}{c_p}\left[\left(\frac{\partial w_s}{\partial T}\right)_p \frac{\mathrm{d}T}{\mathrm{d}z} - \left(\frac{\partial w_s}{\partial p}\right)_T \rho g\right]. \tag{27}$$

Solving for $\mathrm{d}T/\mathrm{d}z$, the lapse rate, Γ_s, of an ascending saturated parcel is given by

$$\Gamma_s = -\frac{\mathrm{d}T}{\mathrm{d}z} = \frac{g}{c_p}\frac{1-\rho L_c(\partial w_s/\partial p)_T}{1+(L_c/c_p)(\partial w_s/\partial T)_p} \approx \frac{g}{c_p}\frac{1+L_c w_s/(R_d T)}{1+L_c^2 w_s/(c_p R_v T^2)}. \tag{28}$$

where the last near equality follows with use of the Clausius–Clapeyron relation. The quantity R_d is the gas constant for dry air and R_v is the gas constant for water vapor, and $R_v = R_d/\alpha_w$. The quantity Γ_s is variously called the *pseudoadiabatic* or *moist adiabatic* or *saturated adiabatic* lapse rate, and it is plotted in Fig. 5.

Because g/c_p is the dry adiabatic lapse rate Γ_d, $\Gamma_s < \Gamma_d$, and values of Γ_s are typically around $6\,\mathrm{K\ km}^{-1}$ in the lower atmosphere; however, $\mathrm{d}w_s/\mathrm{d}T$ is an increasing function of T so that Γ_s decreases with increasing temperature and can be as low as $3.5\,\mathrm{K\ km}^{-1}$. For a saturated parcel, the stability conditions analogous to (21) are

$$\text{stability}: \qquad -\frac{\partial \widetilde{T}}{\partial z} < \Gamma_s, \tag{29a}$$

$$\text{instability}: \qquad -\frac{\partial \widetilde{T}}{\partial z} > \Gamma_s. \tag{29b}$$

where $\widetilde{T}$ is the environmental temperature. The observed environmental profile in convecting situations is often a combination of the dry adiabatic and moist adiabatic profiles: an unsaturated parcel that is is unstable by the dry criterion will rise and cool following a dry adiabat, Γ_d, until it becomes saturated at the lifting condensation level, above which it will rise following a saturation adiabat, Γ_s. Such convection will proceed until the atmospheric column is stable and, especially in low latitudes, the lapse rate of the atmosphere is largely determined by such convective processes.

References

Vallis, G. K., Zurita-Gotor, P., Cairns, C. & Kidston, J., 2014. Response of the large-scale structure of the atmosphere to global warming. *Quart. J. Roy. Meteor. Soc.*. 10.1002/qj.2456.