Lecture 10b: Rossby Waves and Surface Winds

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In this our third lecture we stay with the atmosphere and introduce some dynamics. Our first goal is to understand why there are surface winds, and in particular why there are surface westerlies (Fig. 1). A full explanation of this would require a discussion of baroclinic instability and take up a couple of lectures in itself. We'll skip all that and carry out explicit derivations only for the barotropic vorticity equation, with the reader filling in the gaps phenomenologically. We do note that there are westerly winds aloft in the atmosphere because of the thermal wind relation, $f\partial u/\partial z = \partial b/\partial z$, where b is buoyancy which is like temperature. Thus, a temperature gradient between the equator and the pole implies that the zonal wind increases with height. But this doesn't of itself mean that the surface winds are non-zero – we will need momentum fluxes for that. By the same token, momentum fluxes are not needed to have westerly winds aloft.

We begin with a few basic equations.

1 Momentum Equation

The zonally-averaged momentum, in Cartesian geometry has the form

$$\frac{\partial \overline{u}}{\partial t} - (f + \overline{\zeta})\overline{v} = \frac{\partial}{\partial y}\overline{u'v'} + \frac{\partial \tau}{\partial z} \tag{1}$$

where $f = f_0 + \beta y$ In mid-latitudes we usually neglect the mean advection terms ($\overline{\zeta}\overline{v}$ here) which in midlatitudes are small. If we multiply by density and integrate vertically then, in a steady state the terms on the left-hand side both vanish, whence

$$\tau_s = \int_z \rho\overline{u'v'}\,\mathrm{d}z \tag{2}$$

where τ_s is the surface stress, which is roughly proportional to the surface wind: $\tau_s \approx r\overline{u}_s$ where r is a constant. Thus

$$\overline{u}_s \approx \frac{1}{r}\int_z \rho\overline{u'v'}\,\mathrm{d}z. \tag{3}$$

In other words, the surface winds arise because of the eddy convergence of momentum in the atmosphere. Where does this come from? It turns out that it arises from the sphericity of the Earth which gives rise to differential rotation and Rossby waves, as we shall see.

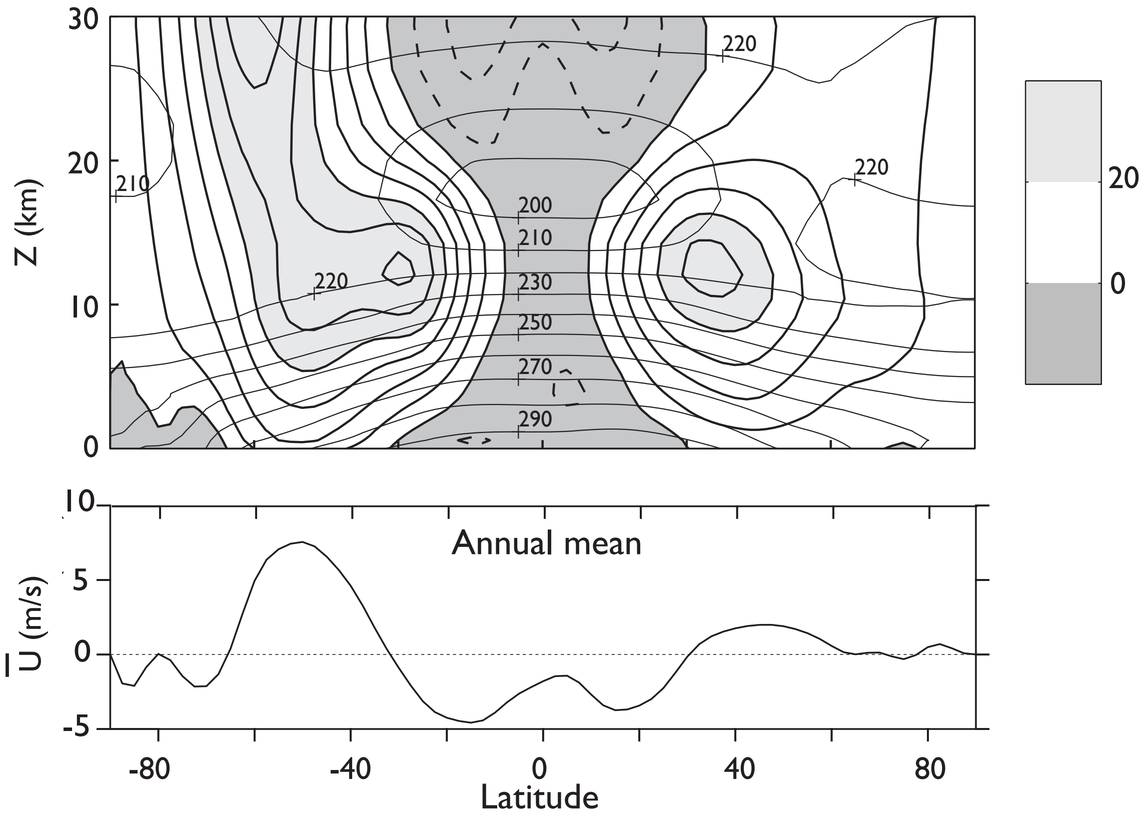

Figure 1: (a) Annual mean, zonally averaged zonal wind (heavy contours and shading) and the zonally averaged temperature (lighter contours). (b) Annual mean, zonally averaged zonal winds at the surface. The wind contours are at intervals of $5\,\mathrm{m\ s}^{-1}$ with shading for eastward winds above $20\,\mathrm{m\ s}^{-1}$ and for all westward winds, and the temperature contours are labelled. The ordinate of (a) and (c) is $Z = -H \log(p/p_R)$, where p_R is a constant, with scale height $H = 7.5\,\mathrm{km}$.

2 Rossby Waves: A Brief Tutorial

The inviscid, adiabatic potential vorticity equation is

$$\frac{\partial q}{\partial t} + \mathbf{u} \cdot \nabla q = 0, \tag{4}$$

where $q(x, y, z, t)$ is the potential vorticity and $\mathbf{u}\,(x, y, z, t)$ is the horizontal velocity. The velocity is related to a streamfunction by $u = -\partial\psi/\partial y$, $v = \partial\psi/\partial x$ and the potential vorticity is some function of the streamfunction, which might differ from system to system. Two examples, one applying to a continuously stratified system and the second to a single layer system, are

$$q = f + \zeta + \frac{\partial}{\partial z}\left(S(z)\frac{\partial \psi}{\partial z}\right), \qquad q = \zeta + f - {k_d}^2\psi. \tag{5a,b}$$

We deal mainly with the second. If the basic state is a zonal flow and purely a function of y then

$$q = \overline{q}(y,z) + q'(x,y,t), \qquad \psi = \overline{\psi}(y,z) + \psi'(x,y,z,t) \tag{6}$$

whence

$$\frac{\partial q'}{\partial t} + \overline{\mathbf{u}} \cdot \nabla \overline{q} + \overline{\mathbf{u}} \cdot \nabla q' + \mathbf{u}' \cdot \nabla \overline{q} + \mathbf{u}' \cdot \nabla q' = 0. \tag{7}$$

Linearizing gives

$$\frac{\partial q'}{\partial t} + \overline{u}\frac{\partial q'}{\partial x} + v'\frac{\partial \overline{q}}{\partial y} = 0. \tag{8}$$

2.1 Rossby waves in a single layer

In the single-layer case we have $q = \beta y + \nabla^2\psi - {k_d}^2\psi$. If we linearize this around a zonal flow then $\psi = \overline{\psi} + \psi'$ and

$$\overline{\psi} = -\overline{u}y \qquad \overline{q} = \beta y + \overline{u}k_d^2 y \tag{9}$$

and

$$q' = \nabla^2\psi' - {k_d}^2\psi' \tag{10}$$

and (8) becomes

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)(\nabla^2\psi' - \psi'{k_d}^2) + \frac{\partial \psi'}{\partial x}(\beta + Uk_d^2) = 0 \tag{11}$$

Substituting $\psi' = \mathrm{Re}\,\widetilde{\psi}\mathrm{e}^{\mathrm{i}(kx+ly-\omega t)}$ we obtain the dispersion relation,

$$\omega = \frac{k(UK^2 - \beta)}{K^2 + {k_d}^2} = Uk - k\frac{\beta + U{k_d}^2}{K^2 + {k_d}^2}. \tag{12}$$

We will simplify by taking $U = 0$ whence

$$\omega = -\frac{\beta}{K^2 + {k_d}^2}. \tag{13}$$

The corresponding components of phase speed and group velocity are

$$c_p^x \equiv \frac{\omega}{k} = -\frac{\beta}{K^2 + {k_d}^2}, \qquad c_p^y \equiv \frac{\omega}{l} = \frac{k}{l}\left(\frac{\beta}{K^2 + {k_d}^2}\right) \tag{14a,b}$$

and

$$c_g^x \equiv \frac{\partial \omega}{\partial k} = \frac{\beta(k^2 - l^2 - {k_d}^2)}{\left(K^2 + {k_d}^2\right)^2}, \qquad c_g^y \equiv \frac{\partial \omega}{\partial l} = \frac{2\beta kl}{\left(K^2 + {k_d}^2\right)^2}, \tag{15a,b}$$

which $K^2 = k^2 + l^2$.

3 Momentum Transport in Rossby Waves

It turns out that Rossby waves will transport momentum from place to place, and this is why we have surface winds! (Well, at least it is an explication of why we have surface winds. Other explications that don't involve Rossby waves can be given (Vallis, 2006), but they are all really the same explanation.)

Let us suppose that some mechanism is present that excites Rossby waves in mid-latitudes. This mechanism is in fact baroclinic instability, but we don't really need to know that. We expect that Rossby waves will be generated there, propagate away and break and dissipate. To the extent that the waves are quasi-linear and do not interact, then just away from the source region each wave has the form

$$\psi = \mathrm{Re}\, C \mathrm{e}^{\mathrm{i}(kx+ly-\omega t)} = \mathrm{Re}\, C \mathrm{e}^{\mathrm{i}(kx+ly-kct)}, \tag{16}$$

where C is a constant, with dispersion relation

$$\omega = ck = \overline{u}k - \frac{\beta k}{k^2+l^2} \equiv \omega_R, \tag{17}$$

taking $k_d = 0$ and provided that there is no meridional shear in the zonal flow. The meridional component of the group velocity is given by

$$c_g^y = \frac{\partial \omega}{\partial l} = \frac{2\beta k l}{(k^2+l^2)^2}. \tag{18}$$

Now, the direction of the group velocity must be *away* from the source region; this is a radiation condition, demanded by the requirement that Rossby waves transport energy *away* from the disturbance. Thus, northwards of the source kl is positive and southwards of the source kl is negative. That the product kl can be positive or negative arises because for each k there are two possible values of l that satisfy the dispersion relation (17), namely

$$l = \pm \left(\frac{\beta}{\overline{u}-c} - k^2 \right)^{1/2}, \tag{19}$$

assuming that the quantity in parentheses is positive.

The velocity variations associated with the Rossby waves are

$$u' = -\mathrm{Re}\, C\, \mathrm{i} l \mathrm{e}^{\mathrm{i}(kx+ly-\omega t)}, \qquad v' = \mathrm{Re}\, C\, \mathrm{i} k \mathrm{e}^{\mathrm{i}(kx+ly-\omega t)}, \tag{20a,b}$$

and the associated momentum flux is

$$\overline{u'v'} = -\frac{1}{2} C^2 k l. \tag{21}$$

Thus, given that the sign of kl is determined by the group velocity, northwards of the source the momentum flux associated with the Rossby waves is southward (i.e., $\overline{u'v'}$ is negative), and southwards of the source the momentum flux is northward (i.e., $\overline{u'v'}$ is positive). That is, the momentum flux associated with the Rossby waves is *toward* the source region. Momentum converges in the region of the stirring, producing net eastward flow there and westward flow to either side (Fig. 2).

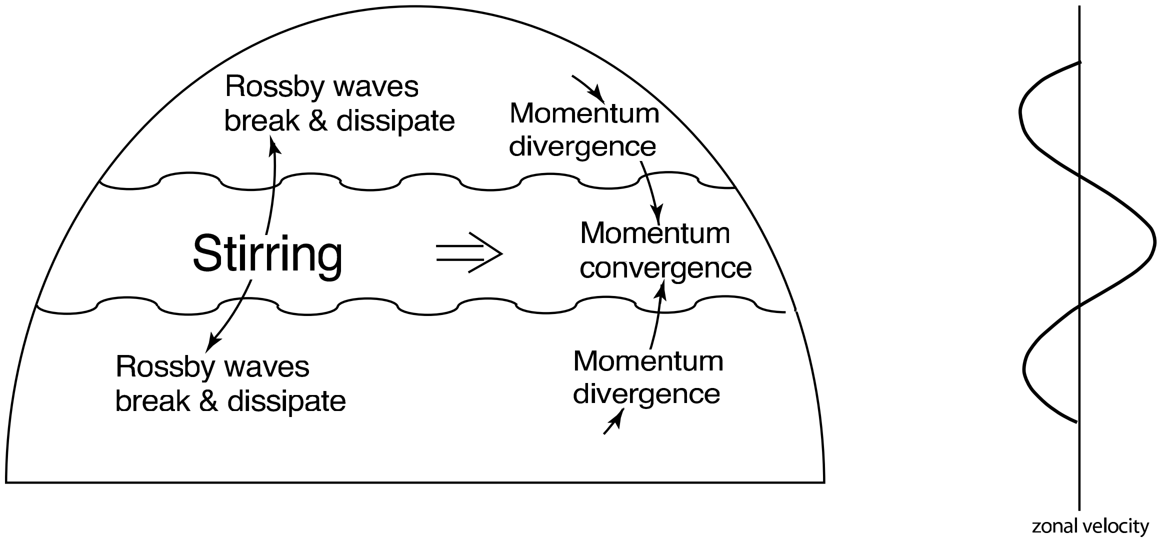

Figure 2: Generation of zonal flow on a β-plane or on a rotating sphere. Stirring in mid-latitudes (by baroclinic eddies) generates Rossby waves that propagate away from the disturbance. Momentum converges in the region of stirring, producing eastward flow there and weaker westward flow on its flanks.

Another way of describing the same effect is to note that if kl is positive then lines of constant phase ($kx + ly =$ constant) are tilted north-west/south-east, as in Fig. 3 and the momentum flux associated with such a disturbance is negative ($\overline{u'v'} < 0$). Similarly, if kl is negative then the constant-phase lines are tilted north-east/south-west and the associated momentum flux is positive ($\overline{u'v'} > 0$). The net result is a convergence of momentum flux into the source region. In physical space this is reflected by having eddies that are shaped like a boomerang, as in Fig. 3.

Pseudomomentum and wave–mean-flow interaction

The kinematic relation between vorticity flux and momentum flux for non-divergent two-dimensional flow is

$$v\zeta = \frac{1}{2}\frac{\partial}{\partial x}\left(v^2 - u^2\right) - \frac{\partial}{\partial y}(uv). \tag{22}$$

After zonal averaging this gives

$$\overline{v'\zeta'} = -\frac{\partial \overline{u'v'}}{\partial y}, \tag{23}$$

noting that $\overline{v} = 0$ for two-dimensional incompressible (or geostrophic) flow.

Now, the barotropic zonal momentum equation is (for horizontally non-divergent flow)

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} - fv = -\frac{\partial \phi}{\partial x} + F_u - D_u, \tag{24}$$

where F_u and D_u represent the effects of any forcing and dissipation. Zonal averaging, with

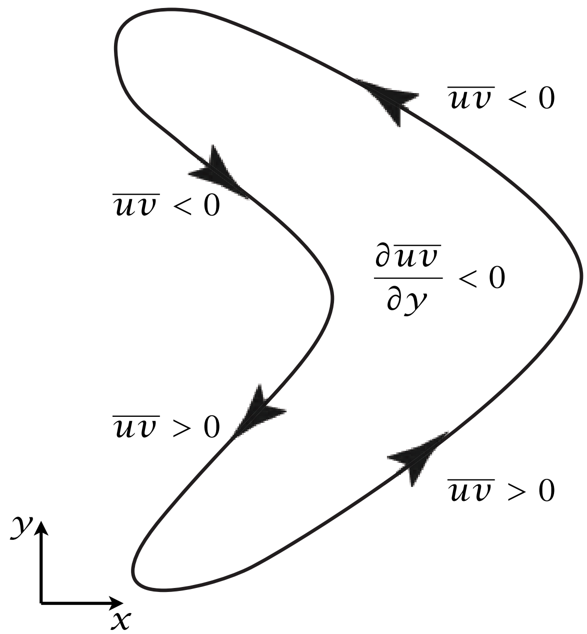

Figure 3: The momentum transport in physical space, caused by the propagation of Rossby waves away from a source in mid-latitudes. The ensuing boomerang-shaped eddies are responsible for a convergence of momentum, as indicated in the idealization pictured.

$\overline{v} = 0$, gives

$$\frac{\partial \overline{u}}{\partial t} = -\frac{\partial \overline{u'v'}}{\partial y} + \overline{F}_u - \overline{D}_u, \tag{25}$$

or, using (23),

$$\frac{\partial \overline{u}}{\partial t} = \overline{v'\zeta'} + \overline{F}_u - \overline{D}_u. \tag{26}$$

Thus, the zonally averaged wind is maintained by the zonally averaged vorticity flux. On average there is little if any direct forcing of horizontal momentum and we may set $\overline{F}_u = 0$, and if the dissipation is parameterized by a linear drag (26) becomes

$$\frac{\partial \overline{u}}{\partial t} = \overline{v'\zeta'} - r\overline{u}, \tag{27}$$

where the constant r is an inverse frictional time scale.

Now consider the maintenance of this vorticity flux. The barotropic vorticity equation is

$$\frac{\partial \zeta}{\partial t} + \mathbf{u} \cdot \nabla \zeta + v\beta = F_\zeta - D_\zeta, \tag{28}$$

where F_ζ and D_ζ are forcing and dissipation of vorticity. Linearize about a mean zonal flow to give

$$\frac{\partial \zeta'}{\partial t} + \overline{u}\frac{\partial \zeta'}{\partial x} + \gamma v' = F'_\zeta - D'_\zeta, \tag{29}$$

where

$$\gamma = \beta - \frac{\partial^2 \overline{u}}{\partial y^2} \tag{30}$$

is the meridional gradient of absolute vorticity. Multiply (29) by ζ'/γ and zonally average, assuming that $\overline{u}_{yy}$ is small compared to β or varies only slowly, to form the pseudomomentum equation,

$$\frac{\partial \mathcal{A}}{\partial t} + \overline{v'\zeta'} = \frac{1}{\gamma}(\overline{\zeta' F'_\zeta} - \overline{\zeta' D'_\zeta}), \tag{31a}$$

$$\mathcal{A} = \frac{1}{2\gamma}\overline{\zeta'^2} \tag{31b}$$

is a wave activity density, equal to the (negative of) the pseudomomentum for this problem. The parameter γ is positive if the average absolute vorticity increases monotonically northwards, and this is usually the case in both Northern and Southern Hemispheres.

3.1 An aside on wave activity and stability

Suppose the flow is unforced and inviscid (common conditions that we impose in stability problems). Then the wave activity equation above becomes

$$\frac{\partial \mathcal{A}}{\partial t} + \overline{v'\zeta'} = 0. \tag{32}$$

This condition holds even in the presence of shear. Integrating between quiescent latitudes gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathcal{A}\,\mathrm{d}y = 0. \tag{33}$$

The quantity $\widehat{A} \equiv \int \mathcal{A}\,\mathrm{d}y$ is wave activity, something that is quadratic in wave amplitude and is conserved. $\mathcal{A}$ itself is a wave activity density. Energy is not normally a wave activity, because it grows if the flow is unstable, whereas a wave activity does not.

Now suppose that γ is positive everywhere. In this case the conservation of $\widehat{A}$ prevents $\overline{\zeta'^2}$ from growing! Thus, for a wave to grow, $\beta - \overline{u}_{yy}$ must change sign somewhere in the domain. We have derived the *Rayleigh-Kuo* criterion for barotropic instability. Note that there is no mention of normal modes, although we have still (in this derivation) assumed linearity.

4 Wave–mean-flow interaction, acceleration and non-acceleration

In the absence of forcing and dissipation, (27) and (31a) imply an important relationship between the change of the mean flow and the pseudomomentum, namely

$$\frac{\partial \overline{u}}{\partial t} + \frac{\partial \mathcal{A}}{\partial t} = 0. \tag{34}$$

We have now essentially derived a special case of the *non-acceleration* result. If the waves are steady and inviscid, then from (31a) $\overline{v'\zeta'} = 0$. Then from (34) the mean flow does not

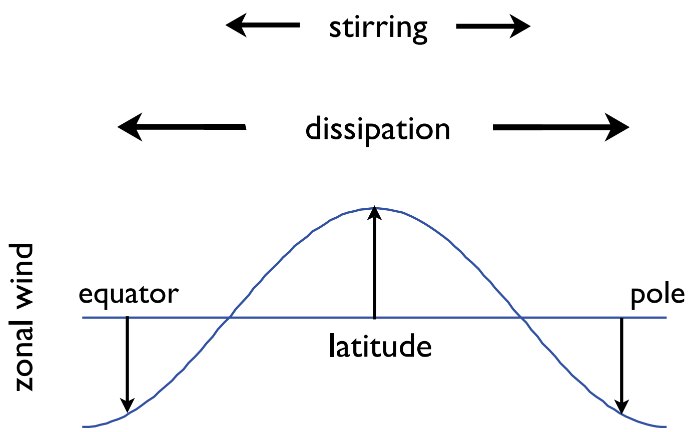

Figure 4: Mean flow generation by a meridionally confined stirring. Because of Rossby wave propagation away from the source region, the distribution of pseudomomentum dissipation is broader than that of pseudomomentum forcing, and the sum of the two leads to the zonal wind distribution shown, with positive (eastward) values in the region of the stirring. See also Fig. 6.

accelerate. We need to do a bit more work in the stratified case, but the essence of the result is the same.

Now if for some reason $\mathcal{A}$ increases, perhaps because a wave enters an initially quiescent region because of stirring elsewhere, then mean flow must decrease. However, because the vorticity flux integrates to zero, the zonal flow cannot decrease everywhere. Thus, if the zonal flow decreases in regions away from the stirring, it must *increase* in the region of the stirring. In the presence of forcing and dissipation this mechanism can lead to the production of a statistically steady jet in the region of the forcing, since (27) and (31a) combine to give

$$\frac{\partial \overline{u}}{\partial t} + \frac{\partial \mathcal{A}}{\partial t} = -r\overline{u} + \frac{1}{\gamma}(\overline{\zeta' F'_\zeta} - \overline{\zeta' D'_\zeta}), \tag{35}$$

and in a statistically steady state

$$r\overline{u} = \frac{1}{\gamma}(\overline{\zeta' F'_\zeta} - \overline{\zeta' D'_\zeta}). \tag{36}$$

The terms on the right-hand side represent the stirring and dissipation of vorticity, and integrated over latitude their sum will vanish, or otherwise the pseudomomentum budget cannot be in a steady state. However, let us suppose that forcing is confined to mid-latitudes. In the forcing region, the first term on the right-hand side of (36) will be larger than the second, and an eastward mean flow will be generated. Away from the direct influence of the forcing, the dissipation term will dominate and westward mean flows will be generated, as sketched in Fig. 4. Thus, *on a β -plane or on the surface of a rotating sphere an eastward mean zonal flow can be maintained by a vorticity stirring that imparts no net momentum to the fluid.* In general, stirring in the presence of a vorticity gradient will give rise to a mean flow, and on a spherical planet the vorticity gradient is provided by differential rotation.

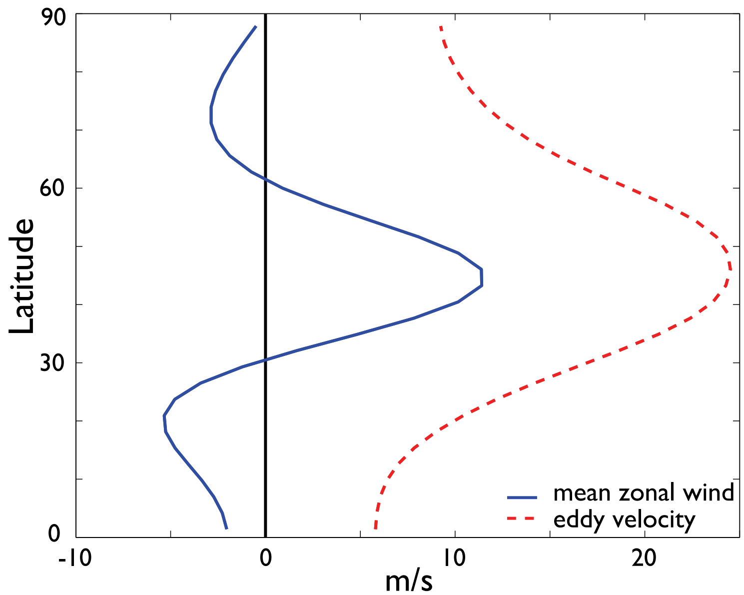

Figure 5: The time and zonally averaged wind (solid line) obtained by an integration of the barotropic vorticity equation on the sphere. The fluid is stirred in mid-latitudes by a random wavemaker that is statistically zonally uniform, acting around zonal wavenumber 8, and that supplies no net momentum. Momentum converges in the stirring region leading to an eastward jet with a westward flow to either side, and zero area-weighted spatially integrated velocity. The dashed line shows the r.m.s. (eddy) velocity created by the stirring.

It is crucial to the generation of a mean flow that the dissipation has a broader latitudinal distribution than the forcing: if all the dissipation occurred in the region of the forcing then from (36) no mean flow would be generated. However, Rossby waves are generated in the forcing region, and these propagate meridionally before dissipating thus broadening the dissipation distribution and allowing the generation of a mean flow.

5 Rossby Waves in an Inhomogeneous Medium

Consider the horizontal problem with infinite deformation radius and linearized equation of motion

$$\left(\frac{\partial}{\partial t} + \overline{u}(y)\frac{\partial}{\partial x}\right) q' + v'\frac{\partial \overline{q}}{\partial y} = 0, \tag{37}$$

where $q' = \nabla^2\psi'$, $v' = \partial\psi'/\partial x$ and $\partial\overline{q}/\partial y = \beta - \overline{u}_{yy}$. If $\overline{u}$ and $\partial\overline{q}/\partial y$ do not vary in space then we may seek wavelike solutions in the usual way and obtain the dispersion relation

$$\omega \equiv ck = \overline{u}k - \frac{\partial\overline{q}/\partial y}{k} k^2 + l^2 \tag{38}$$

where k and l are the x - and y -wavenumbers.

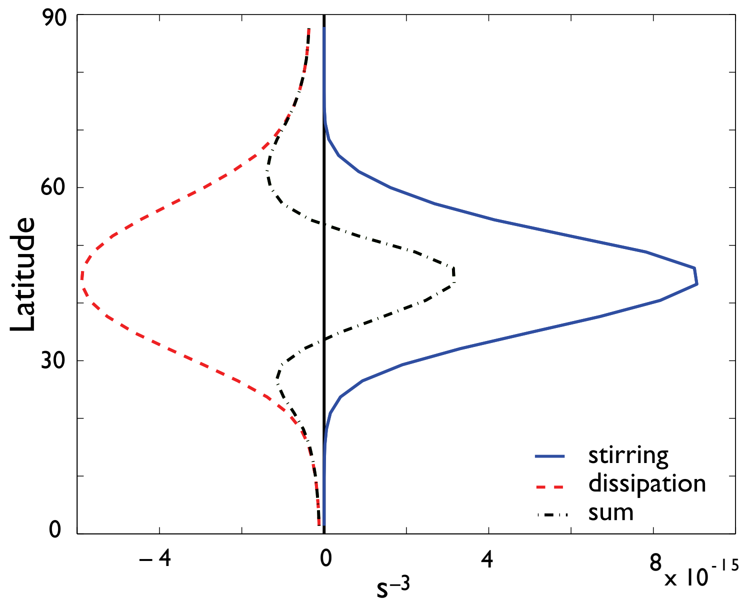

Figure 6: The pseudomomentum stirring (solid line, $\overline{F'_\zeta \zeta'}$), dissipation (dashed line, $\overline{D'_\zeta \zeta'}$) and their sum (dot–dashed), for the same integration as Fig. 5. Because Rossby waves propagate away from the stirred region before breaking, the distribution of dissipation is broader than the forcing, resulting in an eastward jet where the stirring is centred, with westward flow on either side.

If the parameters do vary in the y -direction then we seek a solution of the form $\psi' = \widetilde{\psi}(y)\exp[\mathrm{i}k(x-ct)]$ and obtain

$$\frac{\partial^2 \widetilde{\psi}}{\partial y^2} + l^2(y)\widetilde{\psi} = 0, \quad \text{where} \quad l^2(y) = \frac{\partial \overline{q}/\partial y}{\overline{u} - c} - k^2 \tag{39a,b}$$

If the parameter variation is sufficiently small, occurring on a spatial scale longer than the wavelength of the waves, then we may expect that the disturbance will propagate locally as a plane wave. The solution is then of WKB form namely

$$\widetilde{\psi}(y) = A_0 l^{-1/2} \exp\left(\mathrm{i}\int l\,\mathrm{d}y\right). \tag{40}$$

where A_0 is a constant. The phase of the wave in the y -direction, θ, is evidently given by $\theta = \int l\,\mathrm{d}y$, so that the local wavenumber is given by $\mathrm{d}\theta/\mathrm{d}y = l$. The group velocity is, as before,

$$c_g^x = \overline{u} + \frac{(k^2 - l^2)\partial \overline{q}/\partial y}{(k^2 + l^2)^2}, \qquad c_g^y = \frac{2kl\,\partial \overline{q}/\partial y}{(k^2 + l^2)^2}. \tag{41a,b}$$

The group velocity can now vary spatially, although it is only allowed to vary slowly.

5.1 Wave amplitude

As a Rossby wave propagates its amplitude is not necessarily constant because, in the presence of a shear, the wave may exchange energy with the background state. It goes like

$l^{-1/2}(y)$. This variation can be understood from somewhat more general considerations. As we saw earlier in the simple one-layer case (and discussed more in the appendix) an inviscid, adiabatic wave will conserve its wave activity meaning that

$$\frac{\partial \mathcal{A}}{\partial t} + \nabla \cdot \mathcal{F} = 0, \tag{42}$$

where $\mathcal{A}$ is the wave amplitude and $\mathcal{F}$ is the flux, and $\mathcal{F} = \mathbf{c}_g \mathcal{A}$. In the stratified case we have

$$\mathcal{A} = \frac{\overline{q'^2}}{2\partial\overline{q}/\partial y}, \qquad \mathcal{F} = -\overline{u'v'}\,\mathbf{j} + \frac{f_0}{N^2}\overline{v'b'}\,\mathbf{k}, \tag{43}$$

with $\mathcal{F}$ is the Eliassen–Palm (EP) flux, and in the 2D case there is no buoyancy and the $\mathbf{k}$ component is zero. If the waves are steady then $\nabla \cdot \mathcal{F} = 0$, and in the two-dimensional case under consideration this means that $\partial\overline{u'v'}/\partial y = 0$.

Thus, $u'v' = kl|\widetilde{\psi}|^2 =$ constant, and since k is constant the amplitude of a wave varies like

$$|\widetilde{\psi}| = \frac{A_0}{\sqrt{l(y)}} \tag{44}$$

as in the WKB solution. The energy of the wave then varies like

$$\text{Energy} = (k^2 + l^2)\frac{A_0^2}{l}. \tag{45}$$

6 Rossby Wave Propagation in a Slowly Varying Medium

The linear equation of motion is, in terms of streamfunction,

$$\left(\frac{\partial}{\partial t} + \overline{u}(y,z)\frac{\partial}{\partial x}\right)\left[\nabla^2\psi' + \frac{{f_0}^2}{\rho_R}\frac{\partial}{\partial z}\left(\frac{\rho_R}{N^2}\frac{\partial \psi'}{\partial z}\right)\right] + \frac{\partial \psi'}{\partial x}\frac{\partial \overline{q}}{\partial y} = 0. \tag{46}$$

We suppose that the parameters of the problem vary slowly in y and/or z but are uniform in x and t. The frequency and zonal wavenumber are therefore constant. We seek solutions of the form $\psi' = \widetilde{\psi}(y,z)\mathrm{e}^{\mathrm{i}k(x-ct)}$ and find (if, for simplicity, N^2 and ρ_R are constant)

$$\frac{\partial^2 \widetilde{\psi}}{\partial y^2} + \frac{{f_0}^2}{N^2}\frac{\partial^2 \widetilde{\psi}}{\partial z^2} + n^2(y,z)\widetilde{\psi} = 0 \tag{47a}$$

where

$$n^2(y,z) = \frac{\partial \overline{q}/\partial y}{\overline{u} - c} - k^2. \tag{47b}$$

The value of n^2 must be positive in order that waves can propagate, and so waves cease to propagate when they encounter either

1. *A turning line,* where $n^2 = 0$, or

2. *A critical line,* where $\overline{u} = c$ and n^2 becomes infinite.

The bounds may usefully be expressed as a condition on the zonal flow:

$$0 < \overline{u} - c < \frac{\partial \overline{q}/\partial y}{k^2}. \tag{48}$$

If the length scale over which the parameters of the problem vary is much longer than the wavelengths themselves we can expect the solution to look locally like a plane wave and a WKB analysis can be employed. In the purely horizontal problem we assume a solution of the form $\psi' = \widetilde{\psi}(y)\mathrm{e}^{\mathrm{i}k(x-ct)}$ and find

$$\frac{\partial^2 \widetilde{\psi}}{\partial y^2} + l^2(y)\widetilde{\psi} = 0, \qquad l^2(y) = \frac{\partial \overline{q}/\partial y}{\overline{u} - c} - k^2. \tag{49}$$

The solution is of the form

$$\widetilde{\psi}(y) = A l^{-1/2} \exp\Big(\pm \mathrm{i} \int l \, \mathrm{d}y \Big). \tag{50}$$

Thus, $l(y)$ is the local y -wavenumber, and the amplitude of the solution varies like $l^{-1/2}$. At a critical line the amplitude of the wave will go to zero although the energy may become very large, and since the wavelength is small the waves may break. At a turning line the amplitude and energy will both be large, but since the wavelength is long the waves will not necessarily break. A similar analysis may be employed for vertically propagating Rossby waves.

6.1 Two examples

(i) Waves with a turning latitude

A turning line arises where $l = 0$. The line arises if the potential vorticity gradient diminishes to such an extent that $l^2 < 0$ and the waves then cease to propagate in the y -direction. This may happen even in unsheared flow as a wave propagates polewards and the magnitude of beta diminishes.

As a wave packet approaches a turning latitude then l goes to zero so the amplitude, and the energy, of the wave approach infinity. This may happen as a wave propagates polewards and β diminishes. However, the wave will never reach the turning latitude because the meridional component of the group velocity is zero, as can be seen from the expressions for the group velocity, (41). As a wave approaches the turning latitude $c_g^x \to (\beta - \overline{u}_{yy})/k^2$ and $c_g^y \to 0$, so the group velocity is purely zonal and indeed as $l \to 0$

$$\frac{c_g^x - \overline{u}}{c_g^y} = \frac{k}{2l} \to \infty. \tag{51}$$

Because the meridional wavenumber is small the wavelength is large, so we do not expect the waves to break. Rather, we intuitively expect that a wave packet will turn — hence the eponym 'turning latitude' — and be reflected.

To illustrate this, consider waves propagating in a background state that has a beta effect that diminishes polewards but no horizontal shear. To be concrete suppose that $\beta = 5$ at $y = 0$, diminishing linearly to $\beta = 0$ at $y = 0$, and that $\overline{u} - c = 1$ everywhere.

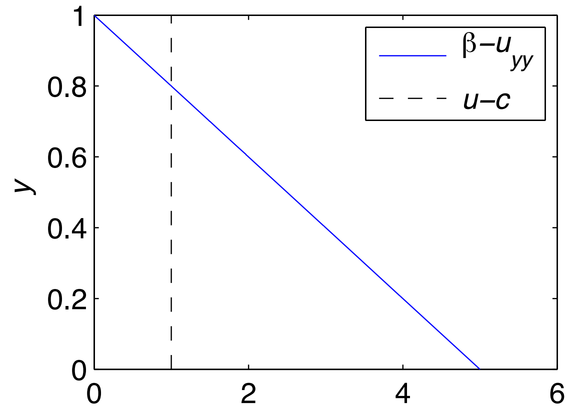

Figure 7: Parameters for the first example considered in section 6.1, with all variables nondimensional. The zonal flow is uniform with $u = 1$ and $c = 0$ (so that $\overline{u}_{yy} = 0$) and β diminishes linearly as y increases polewards as shown. With zonal wavenumber $k = 1$ there is a turning latitude at $y = 0.8$, and the wave properties are illustrated in Fig. 8.

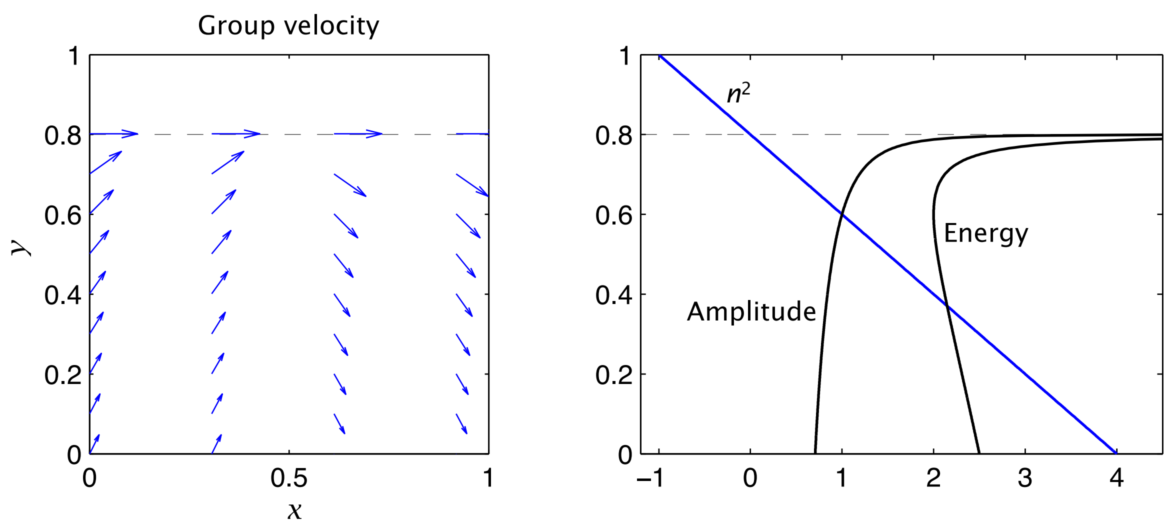

Figure 8: Left: The group velocity evaluated using (41) for the parameters illustrated in Fig. 7, which give a turning latitude at $y = 0.8$. For $x < 0.5$ we choose positive values of n, and a northward group velocity, whereas for $x > 0.5$ we choose negative values of n. Right panel: Values of refractive index squared (n^2), the energy and the amplitude of a wave. n^2 is negative for $y > 0.8$. See text for more description.

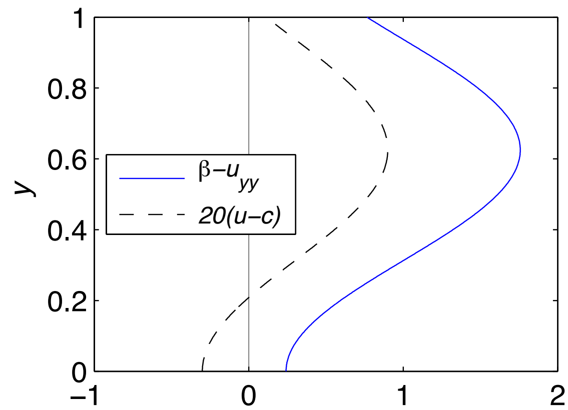

Figure 9: Parameters for the second example considered in section 6.1, with all variables nondimensional. The zonal flow has a broad eastward jet and β is constant. There is a critical line at $y = 0.2$, and with zonal wavenumber $k = 5$ the wave properties are illustrated in Fig. 10.

There is no critical line but depending on the x -wavenumber there may be a turning line, and if we choose $k = 1$ then the turning line occurs when $\beta = 1$ and so at $y = 0.8$. Note that the turning latitude depends on the value of the x -wavenumber — if the zonal wavenumber is larger then waves will turn further south. The parameters are illustrated in Fig. 7.

For a given zonal wavenumber ($k = 1$ in this example) the value of l^2 is computed using (39b), and the components of the group velocity using (41), and these are illustrated in Fig. 8. Note that we may choose either a positive or a negative value of l, corresponding to northward or southward oriented waves, and we illustrate both in the figure. The value of l^2 becomes zero at $y = 0.8$, and this corresponds to a turning latitude. The values of the wave amplitude and energy are computed using (44) and (45) (with an arbitrary amplitude at $y = 0$) and these both become infinite at the turning latitude.

(ii) Waves with a critical latitude

A critical line occurs when $\overline{u} = c$, corresponding to the upper bound of c, and from (39) we see that at a critical line the meridional wavenumber approaches infinity. From (41) we see that both the x - and y -components of the group velocity are zero — a wave packet approaching a critical line just stops. Specifically, as l becomes large

$$c_g^x - \overline{u} \to 0, \qquad c_g^y \to 0, \qquad \frac{c_g^x - \overline{u}}{c_g^y} \to -\frac{l}{k} \to -\infty. \tag{52}$$

From (44) the amplitude of the wave packet also approaches zero, but its energy approaches infinity. Since the wavelength is very small we expect the waves to *break* and deposit their momentum, and this situation commonly arises when Rossby waves excited in midlatitudes propagate equatorward and encounter a critical latitude in the subtropics.

To illustrate this let us construct background state that has an eastward jet in midlatitudes becoming westward at low latitudes, with β constant chosen to be large enough so that

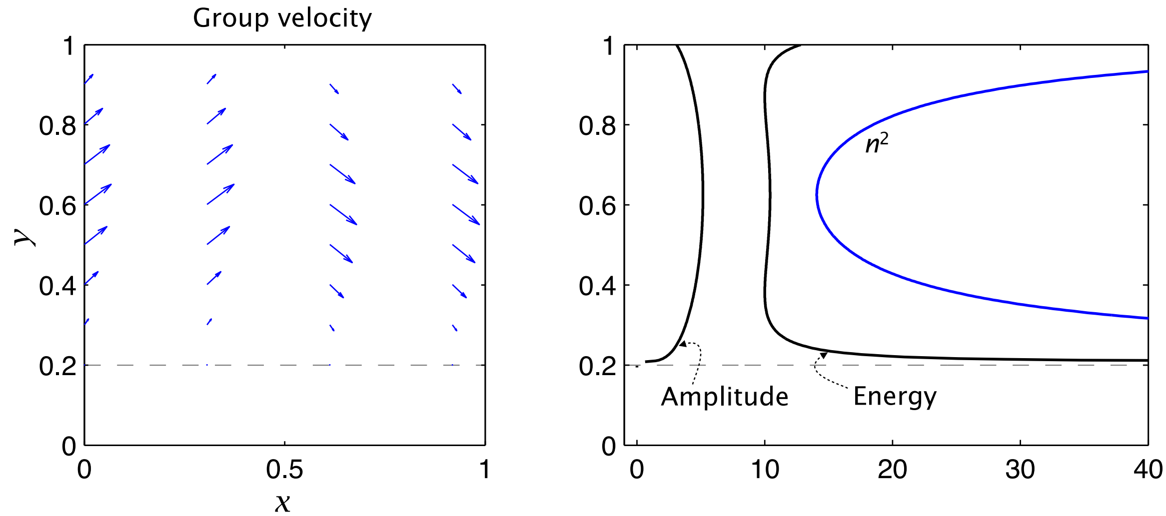

Figure 10: Left: The group velocity evaluated using (41) for the parameters illustrated in Fig. 7, which give a critical line at $y = 0.2$. For $x < 0.5$ we choose positive values of n, and a northward group velocity, whereas for $x > 0.5$ we choose negative values of n. Right panel: Values of refractive index squared, the energy and the amplitude of a wave. The value of n^2 becomes infinite at the critical line. See text for more description.

$\beta - \overline{u}_{yy}$ is positive everywhere. (Specifically, we choose $\beta = 1$ and $\overline{u} = -0.03\sin(8\pi y/5 + \pi/2) - 0.5)$, but the precise form is not important.) If $c = 0$ then there is a critical line when $\overline{u}$ passes through zero, which in this example occurs at $x = 0.2$. (The value of $\overline{u} - c$ is small at $y = 1$, but no critical line is actually reached.) These parameters are illustrated in Fig. 9. We also choose $k = 5$, which results in a positive value for l^2 everywhere.

As in the previous example, we compute the value of l^2 using (39b) and the components of the group velocity using (41), and these are illustrated in Fig. 10, with northward propagating waves shown for $x < 0.5$ and southward propagating waves for $x > 0.5$. The value of l^2 increases considerably at the northern and southern edges of the domain, and is actually infinite at the critical line at $y = 0.2$. Using (44) the amplitude of the wave diminishes as the critical line approaches, but the energy increases rapidly, suggesting that the linear approximation will break down. The waves will actually stall before reaching the critical layer, because both the x and the y components of the group velocity become very small. Also, because the wavelength is so small we may expect the waves to break and deposit their momentum, but a full treatment of waves in the vicinity of a critical layer requires a nonlinear analysis.

The situation illustrated in this example is of particular relevance to the maintenance of the zonal wind structure in the troposphere. Waves are generated in midlatitude and propagate equatorward and on encountering a critical layer in the subtropics they break, deposit westward momentum and retard the flow, as the reader who braves the next section will discover explicitly.

7 Rossby Wave Absorption near a Critical Layer

We noted in the last section that as a wave approaches a critical latitude the meridional wavenumber l becomes very large, but the group velocity itself becomes small. These observations suggest that the effects of friction might become very large and that the wave would deposit its momentum, thereby accelerating or decelerating the mean flow, and if we are willing to make one or two approximations we can construct an explicit analytic model of this phenomena. Specifically, we will need to choose a simple form for the friction and assume that the background properties vary slowly, so that we can use a WKB approximation. Note that we have to include some form of dissipation, otherwise the Eliassen–Palm flux divergence is zero and there is no momentum deposition by the waves.

7.1 A model problem

Consider horizontally propagating Rossby waves obeying the linear barotropic vorticity equation on the beta-plane (vertically propagating waves may be considered using similar techniques). The equation of motion is

$$\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)\nabla^2\psi+\beta^*\frac{\partial \psi}{\partial x}=-r\nabla^2\psi, \tag{53}$$

where $\beta^*=\beta-\overline{u}_{yy}$. The parameter r is a drag coefficient that acts directly on the relative vorticity. It is not a particularly realistic form of dissipation but its simplicity will serve our purpose well. We shall assume that r is small compared to the Doppler-shifted frequency of the waves and seek solutions of the form

$$\psi'(x,y,t)=\widetilde{\psi}(y)\mathrm{e}^{\mathrm{i}(k(x-ct))}. \tag{54}$$

Substituting into (53) we find, after a couple of lines of algebra, that $\widetilde{\psi}$ satisfies, analogously to (39),

$$\frac{\partial^2\widetilde{\psi}}{\partial y^2}+l^2(y)\widetilde{\psi}=0, \quad \text{where} \quad l^2(y)=\frac{\beta^*}{\overline{u}-c-\mathrm{i}r/k}-k^2. \tag{55a,b}$$

Evidently, as with the inviscid case, if the zonal wind has a lateral shear then l is a function of y. However, l now has an imaginary component so that the wave decays away from its source region. We can already see that if $\overline{u}=c$ the decay will be particularly strong.

7.2 WKB solution

Let us suppose that the zonal wavenumber is small compared to the meridional wavenumber l, which will certainly be the case approaching a critical layer. If $r \ll k(\overline{u}-c)$ then the meridional wavenumber is given by

$$l^2(y)\approx\left[\frac{\beta^*(\overline{u}-c+\mathrm{i}r/k)}{(\overline{u}-c)^2+r^2/k^2}\right]\approx\frac{\beta^*}{\overline{u}-c}\left[1+\frac{\mathrm{i}r}{k(\overline{u}-c)}\right] \tag{56}$$

whence

$$l(y)\approx\left(\frac{\beta^*}{\overline{u}-c}\right)^{1/2}\left[1+\frac{\mathrm{i}r}{2k(\overline{u}-c)}\right]. \tag{57}$$

The streamfunction itself is then given by, in the WKB approximation,

$$\widetilde{\psi} = A l^{-1/2} \exp\left(\pm \mathrm{i} \int^{y} l \, \mathrm{d}y'\right). \tag{58}$$

But now the wave will decay as it moves away from its source and deposit momentum into the mean flow, as we now calculate.

The momentum flux, F_k, associated with the wave with x -wavenumber of k is given by

$$F_k(y) = \overline{u'v'} = -\mathrm{i}k\left(\psi \frac{\partial \psi^*}{\partial y} - \psi^* \frac{\partial \psi}{\partial y}\right), \tag{59}$$

and using (57) and (58) in (59) we obtain

$$F_k(y) = F_0 \exp\left(\pm \mathrm{i} \int_0^{y} (l - l^*) \, \mathrm{d}y'\right) = F_0 \exp\left(\int_0^{y} \frac{\pm r \beta^{*1/2}}{k(\overline{u} - c)^{3/2}} \, \mathrm{d}y'\right). \tag{60}$$

In deriving this expression we use that fact that the amplitude of $\widetilde{\psi}$ (i.e., $l^{-1/2}$) varies only slowly with y so that when calculating $\partial\widetilde{\psi}/\partial y$ its derivative may be ignored. In (60) F_0 is the value of the flux at $y = 0$ and the sign of the exponent must be chosen so that the group velocity is directed away from the wave source region. Clearly, if $r = 0$ then the momentum flux is constant.

The integrand in (60) is the attenuation rate of the wave and it has a straightforward physical interpretation. Using the real part of (57) in (41b), and assuming $|l| \gg |k|$, the meridional component of the group velocity is given by

$$c_g^y = \frac{2kl\,\beta^*}{(k^2 + l^2)^2} \approx \frac{2k\,\beta^*}{l^3} = \frac{2k(\overline{u} - c)^{3/2}}{\beta^{*1/2}}. \tag{61a,b}$$

Thus, we have

$$\text{Wave attenuation rate} = \frac{r\beta^{*1/2}}{k(\overline{u} - c)^{3/2}} = \frac{2 \times \text{Dissipation rate} = 2r}{\text{Meridional group velocity}, c_g^y}. \tag{62}$$

As the group velocity diminishes the dissipation has more time to act and so the wave is preferentially attenuated, a result that we discuss more in the next subsection.

How does this attenuation affect the mean flow? The mean flow is subject to many waves and so obeys the equation

$$\frac{\partial \overline{u}}{\partial t} = -\sum_k \frac{\partial F_k}{\partial y} + \text{viscous terms}. \tag{63}$$

Because the amplitude varies only slowly compared to the phase, the amplitude of $\partial F_k/\partial y$ varies mainly with the attenuation rate (62) and is largest near a critical layer. Consider a Rossby wave propagating away from some source region with a given frequency and x -wavenumber. Because k is negative a Rossby wave always carries westward (or negative) momentum with it. That is, F_k is always negative and increases (becomes more positive) as the wave is attenuated; that is to say, if $r \neq 0$ then $\partial F_k/\partial y$ is positive and from (63) the mean flow is accelerated *westward* as the wave dissipates. This acceleration will be particularly strong if the wave approaches a critical layer where $\overline{u} = c$. Indeed, such a situation arises when Rossby waves, generated in mid-latitudes, propagate equatorward. As the waves enter the subtropics $\overline{u} - c$ becomes smaller and the waves dissipate, producing a westward force on the mean flow, even though a true critical layer may never be reached. Globally, momentum is conserved because there is an equal and opposite (and therefore eastward) wave force at the wave source producing an eddy-driven jet, as discussed in the previous lecture.

7.3 Interpretation using wave activity

We can derive and interpret the above results by thinking about the propagation of wave activity. For barotropic Rossby waves, multiply (53) by ζ/β^* and zonally average to obtain the wave activity equation,

$$\frac{\partial \mathcal{A}}{\partial t} + \frac{\partial \mathcal{F}}{\partial y} = -\alpha \mathcal{A}, \tag{64}$$

where $\mathcal{A} = \overline{\zeta'^2}/2\beta^*$ is the wave activity density, $\partial\mathcal{F}/\partial y = \overline{v'\zeta'}$ is its flux divergence, and $\alpha = 2r$. Referring as needed to the discussion in sections A.2 and A.3, the flux obeys the group velocity property so that

$$\frac{\partial \mathcal{A}}{\partial t} + \frac{\partial}{\partial y}(\mathbf{c}_g \mathcal{A}) = -\alpha \mathcal{A}. \tag{65}$$

Let us suppose that the wave is in a statistical steady state and that the spatial variation of the group velocity occurs on a longer spatial scale than the variations in wave activity density, consistent with the WKB approximation. We then have

$$c_g^y \frac{\partial \mathcal{A}}{\partial y} = -\alpha \mathcal{A}. \tag{66}$$

which integrates to give

$$\mathcal{A}(y) = \mathcal{A}_0 \exp\left(-\int^y \frac{\alpha}{c_g^y}\, \mathrm{d}y'\right). \tag{67}$$

That is, the attenuation rate of the wave activity is the dissipation rate of wave activity divided by the group velocity, as in (60) and (62) (note that $\alpha = 2r$).

Appendix A: Various properties of Rossby Waves

In this appendix we derive various properties of Rossby waves useful in wave–mean-flow interaction theory, assuming a good knowledge of stratified quasi-geostrophic theory. We use the Boussinesq approximation throughout. This material was not presented in the lectures at Walsh.

A.1 The Eliassen–Palm Flux

The eddy flux of potential vorticity may be expressed in terms of vorticity and buoyancy fluxes as

$$v'q' = v'\zeta' + f_0\, v' \frac{\partial}{\partial z}\left(\frac{b'}{N^2}\right). \tag{68}$$

The second term on the right-hand side can be written as

$$\begin{aligned} f_0\, v' \frac{\partial}{\partial z}\left(\frac{b'}{N^2}\right) &= f_0\, \frac{\partial}{\partial z}\left(\frac{v'b'}{N^2}\right) - f_0\, \frac{\partial v'}{\partial z}\frac{b'}{N^2} \\ &= f_0\, \frac{\partial}{\partial z}\left(\frac{v'b'}{N^2}\right) - f_0\, \frac{\partial}{\partial x}\left(\frac{\partial \psi'}{\partial z}\right)\frac{b'}{N^2} \\ &= f_0\, \frac{\partial}{\partial z}\left(\frac{v'b'}{N^2}\right) - \frac{{f_0}^2}{2N^2}\frac{\partial}{\partial x}\left(\frac{\partial \psi'}{\partial z}\right)^2, \end{aligned} \tag{69}$$

using $b' = f_0\, \partial\psi'/\partial z$.

Similarly, the flux of relative vorticity can be written

$$v'\zeta' = -\frac{\partial}{\partial y}(u'v') + \frac{1}{2}\frac{\partial}{\partial x}(v'^2 - u'^2) \tag{70}$$

Using (69) and (70), (68) becomes

$$v'q' = -\frac{\partial}{\partial y}(u'v') + \frac{\partial}{\partial z}\left(\frac{f_0}{N^2}v'b'\right) + \frac{1}{2}\frac{\partial}{\partial x}\left((v'^2 - u'^2) - \frac{b'^2}{N^2}.\right) \tag{71}$$

Thus the meridional potential vorticity flux, in the quasi-geostrophic approximation, can be written as the divergence of a vector: $v'q' = \nabla \cdot \mathcal{E}$ where

$$\mathcal{E} \equiv \frac{1}{2}\left((v'^2 - u'^2) - \frac{b'^2}{N^2}\right)\mathbf{i} - (u'v')\,\mathbf{j} + \left(\frac{f_0}{N^2}v'b'\right)\mathbf{k}. \tag{72}$$

A particularly useful form of this arises after zonally averaging, for then (71) becomes

$$\overline{v'q'} = -\frac{\partial}{\partial y}\overline{u'v'} + \frac{\partial}{\partial z}\left(\frac{f_0}{N^2}\overline{v'b'}\right). \tag{73}$$

The vector defined by

$$\mathcal{F} \equiv -\overline{u'v'}\,\mathbf{j} + \frac{f_0}{N^2}\overline{v'b'}\,\mathbf{k} \tag{74}$$

is called the (quasi-geostrophic) *Eliassen–Palm (EP) flux* (Eliassen & Palm (1961)), and its divergence, given by (73), gives the poleward flux of potential vorticity:

$$\overline{v'q'} = \nabla_x \mathcal{F}, \tag{75}$$

where $\nabla_x \equiv (\partial/\partial y, \partial/\partial z)\cdot$ is the divergence in the meridional plane. Unless the meaning is unclear, the subscript x on the meridional divergence will be dropped.

A.2 The Eliassen–Palm relation

On dividing by $\partial\overline{q}/\partial y$ and using (75), the enstrophy equation becomes

$$\frac{\partial \mathcal{A}}{\partial t} + \nabla \cdot \mathcal{F} = \mathcal{D}, \tag{76a}$$

where

$$\mathcal{A} = \frac{\overline{q'^2}}{2\partial\overline{q}/\partial y}, \qquad \mathcal{D} = \frac{\overline{D'q'}}{\partial\overline{q}/\partial y}. \tag{76b}$$

Equation (76a) is known as the *Eliassen–Palm relation,* and it is a conservation law for the *wave activity density* $\mathcal{A}$. The conservation law is exact (in the linear approximation) if the mean flow is constant in time. It will be a good approximation if $\partial\overline{q}/\partial y$ varies slowly compared to the variation of $\overline{q'^2}$.

If we integrate (76b) over a meridional area A bounded by walls where the eddy activity vanishes, and if $\mathcal{D} = 0$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_A \mathcal{A}\,\mathrm{d}A = 0. \tag{77}$$

The integral is a wave activity — a quantity that is quadratic in the amplitude of the perturbation and that is conserved in the absence of forcing and dissipation. In this case $\mathcal{A}$ is the negative of the *pseudomomentum,* for reasons we will encounter later. ('Wave action' is a particular form of wave activity; it is the energy divided by the frequency and it is a conserved property in many wave problems.) Note that neither the perturbation energy nor the perturbation enstrophy are wave activities of the linearized equations, because there can be an exchange of energy or enstrophy between mean and perturbation — indeed, this is how a perturbation grows in baroclinic or barotropic instability! This is already evident from an enstrophy equation. Or, in general, take the linearized PV equation with $D' = 0$ and multiply by q' to give the enstrophy equation

$$\frac{1}{2}\frac{\partial q'^2}{\partial t} + \frac{1}{2}\overline{\mathbf{u}} \cdot \nabla q'^2 + \mathbf{u}'q' \cdot \nabla\overline{q} = 0, \tag{78}$$

where here the overbar is an average (although it need not be a zonal average). Integrating this over a volume V gives

$$\frac{\mathrm{d}\hat{Z}'}{\mathrm{d}t} \equiv \frac{\mathrm{d}}{\mathrm{d}t}\int_V \frac{1}{2}q'^2\,\mathrm{d}V = -\int_V \mathbf{u}'q' \cdot \nabla\overline{q}\,\mathrm{d}V. \tag{79}$$

The right-hand side does not, in general, vanish and so $\hat{Z}'$ is not in general conserved.

A.3 The group velocity property for Rossby waves

The vector $\mathcal{F}$ describes how the wave activity propagates. In the case in which the disturbance is composed of plane or almost plane waves that satisfy a dispersion relation, then $\mathcal{F} = \mathbf{c}_g\mathcal{A}$, where $\mathbf{c}_g$ is the group velocity and (76a) becomes

$$\frac{\partial \mathcal{A}}{\partial t} + \nabla \cdot (\mathcal{A}\mathbf{c}_g) = 0. \tag{80}$$

This is a useful property, because if we can diagnose $\mathbf{c}_g$ from observations we can use (76a) to determine how wave activity density propagates. Let us demonstrate this explicitly for the pseudomomentum in Rossby waves, that is for (76a).

The Boussinesq quasi-geostrophic equation on the β -plane, linearized around a uniform zonal flow and with constant static stability, is

$$\frac{\partial q'}{\partial t} + \overline{u}\frac{\partial q'}{\partial x} + v'\frac{\partial \overline{q}}{\partial y} = 0, \tag{81}$$

where $q' = [\nabla^2 + (f_0{}^2/N^2)\partial^2/\partial z^2]\psi'$ and, if $\overline{u}$ is constant, $\partial\overline{q}/\partial y = \beta$. Thus, we have

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)\left[\nabla^2\psi' + \frac{\partial}{\partial z}\left(\frac{f_0{}^2}{N^2}\frac{\partial \psi'}{\partial z}\right)\right] + \beta\frac{\partial \psi'}{\partial x} = 0. \tag{82}$$

Seeking solutions of the form

$$\psi' = \mathrm{Re}\,\widetilde{\psi}\mathrm{e}^{\mathrm{i}(kx+ly+mz-\omega t)}, \tag{83}$$

we find the dispersion relation,

$$\omega = \overline{u}k - \frac{\beta k}{\kappa^2}. \tag{84}$$

where $\kappa^2 = (k^2 + l^2 + m^2 f_0{}^2/N^2)$, and the group velocity components:

$$c_g^y = \frac{2\beta kl}{\kappa^4}, \qquad c_g^z = \frac{2\beta km f_0{}^2/N^2}{\kappa^4}. \tag{85}$$

Also, if $u' = \mathrm{Re}\,\widetilde{u}\exp[\mathrm{i}(kx+ly+mz-\omega t)]$, and similarly for the other fields, then

$$\begin{aligned} \widetilde{u} &= -\mathrm{Re}\ \mathrm{i}l\widetilde{\psi}, \qquad & \widetilde{v} &= \mathrm{Re}\ \mathrm{i}k\widetilde{\psi}, \\ \widetilde{b} &= \mathrm{Re}\,\mathrm{i}mf_0\,\widetilde{\psi}, \qquad & \widetilde{q} &= -\mathrm{Re}\,\kappa^2\widetilde{\psi}, \end{aligned} \tag{86}$$

The wave activity density is then

$$\mathcal{A} = \frac{1}{2}\frac{\overline{q'^2}}{\beta} = \frac{\kappa^4}{4\beta}|\widetilde{\psi}^2|, \tag{87}$$

where the additional factor of 2 in the denominator arises from the averaging. Using (86) the EP flux, (74), is

$$\mathcal{F}^y = -\overline{u'v'} = \frac{1}{2}kl|\widetilde{\psi}^2|, \qquad \mathcal{F}^z = \frac{f_0}{N^2}\overline{v'b'} = \frac{f_0{}^2}{2N^2}km|\widetilde{\psi}^2|. \tag{88}$$

Using (85), (87) and (88) we obtain

$$\mathcal{F} = (\mathcal{F}^y, \mathcal{F}^z) = \mathbf{c}_g\mathcal{A}. \tag{89}$$

If the properties of the medium are slowly varying, so that a (spatially varying) group velocity can still be defined, then this is a useful expression to estimate how the wave activity propagates in the atmosphere and in numerical simulations.

A.4 Energy flux in Rossby waves

Start with

$$\frac{\partial}{\partial t}\left(\nabla^2 - k_d^2\right)\psi + \beta\frac{\partial\psi}{\partial x} = 0. \tag{90}$$

To obtain an energy equation multiply (90) by $-\psi$ and obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\left((\nabla\psi)^2 + k_d^2\psi^2\right) - \nabla\cdot\left(\psi\nabla\frac{\partial\psi}{\partial t} + \mathbf{i}\frac{\beta}{2}\psi^2\right) = 0, \tag{91}$$

where $\mathbf{i}$ is the unit vector in the x direction. The first group of terms are the energy itself, or more strictly the energy density. (An energy density is an energy per unit mass or per unit volume, depending on the context.) The term $(\nabla\psi)^2/2 = (u^2+v^2)/2$ is the kinetic energy and $k_d^2\psi^2/2$ is the potential energy, proportional to the displacement of the free surface, squared. The second term is the energy flux, so that we may write

$$\frac{\partial E}{\partial t} + \nabla\cdot\mathbf{F} = 0. \tag{92}$$

where $E = (\nabla\psi)^2/2 + k_d{}^2\psi^2$ and $\mathbf{F} = -\left(\psi\nabla\partial\psi/\partial t + \mathbf{i}\beta\psi^2\right)$. We haven't yet used the fact that the disturbance has a dispersion relation, and if we do so we may expect that the energy moves at the group velocity. Let us now demonstrate this explicitly.

We assume a solution of the form

$$\psi = A(x)\cos(\mathbf{k}\cdot\mathbf{x} - \omega t) = A(x)\cos\left(kx + ly - \omega t\right) \tag{93}$$

where $A(x)$ is assumed to vary slowly compared to the nearly plane wave. (Note that $\mathbf{k}$ is the wave vector, to be distinguished from $\mathbf{k}$, the unit vector in the z -direction.) The kinetic energy in a wave is given by

$$KE = \frac{A^2}{2}\left(\psi_x^2 + \psi_y^2\right) \tag{94}$$

so that, averaged over a wave period,

$$\overline{KE} = \frac{A^2}{2}(k^2+l^2)\frac{\omega}{2\pi}\int_0^{2\pi/\omega}\sin^2(\mathbf{k}\cdot\mathbf{x} - \omega t)\,\mathrm{d}t. \tag{95}$$

The time-averaging produces a factor of one half, and applying a similar procedure to the potential energy we obtain

$$\overline{KE} = \frac{A^2}{4}(k^2+l^2), \qquad \overline{PE} = \frac{A^2}{4}k_d^2, \tag{96}$$

so that the average total energy is

$$\overline{E} = \frac{A^2}{4}(K^2 + k_d{}^2), \tag{97}$$

where $K^2 = k^2 + l^2$.

The flux, $\mathbf{F}$, is given by

$$\mathbf{F} = -\left(\psi\nabla\frac{\partial\psi}{\partial t} + \mathbf{i}\frac{\beta}{2}\psi^2\right) = -A^2\cos^2(\mathbf{k}\cdot\mathbf{x} - \omega t)\left(\mathbf{k}\omega - \mathbf{i}\frac{\beta}{2}\right), \tag{98}$$

so that evidently the energy flux has a component in the direction of the wavevector, $\mathbf{k}$, and a component in the x -direction. Averaging over a wave period straightforwardly gives us additional factors of one half:

$$\overline{\mathbf{F}} = -\frac{A^2}{2}\left(\mathbf{k}\omega + \mathbf{i}\frac{\beta}{2}\right). \tag{99}$$

We now use the dispersion relation $\omega = -\beta k/(K^2 + k_d{}^2)$ to eliminate the frequency, giving

$$\overline{\mathbf{F}} = \frac{A^2\beta}{2}\left(\mathbf{k}\frac{k}{K^2 + k_d{}^2} - \mathbf{i}\frac{1}{2}\right), \tag{100}$$

and writing this in component form we obtain

$$\overline{\mathbf{F}} = \frac{A^2\beta}{4}\left[\mathbf{i}\left(\frac{k^2 - l^2 - k_d{}^2}{K^2 + k_d{}^2}\right) + \mathbf{j}\left(\frac{2kl}{K^2 + k_d^2}\right)\right] \tag{101}$$

Comparison of (101) with (15) and (97) reveals that

$$\overline{\mathbf{F}} = \mathbf{c}_g\overline{E} \tag{102}$$

so that the energy propagation equation, (92), when averaged over a wave, becomes

$$\frac{\partial\overline{E}}{\partial t} + \nabla\cdot\mathbf{c}_g\overline{E} = 0. \tag{103}$$

This is an important result, and more general than our derivation implies. One immediate implication is that if there is a disturbance that generates waves, *the group velocity is directed away from the disturbance.*

Most of the time in waves, energy is not conserved because it can be extracted from the flow.

Appendix B: The WKB Approximation for Linear Waves

We are concerned with finding solutions to an equation of the form

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}z^2} + m^2(z)\xi = 0, \tag{104}$$

where $m^2(z)$ is positive for wavelike solutions. If m is constant the solution has the harmonic form

$$\xi = \mathrm{Re}\, A_0\mathrm{e}^{imz} \tag{105}$$

where A_0 is a complex constant. If m varies only 'slowly' with z — meaning that the variations occur on a scale much longer than $1/m$ — one might reasonably expect that the harmonic solution above would provide a reasonable first approximation; that is, we expect the solution to locally look like a plane wave with local wavenumber $m(z)$. However, we might also expect that the solution would not be *exactly* of the form $\exp(\mathrm{i}m(z)z)$, because the phase of ξ is $\theta(z) = mz$, so that $\mathrm{d}\theta/\mathrm{d}z = m + z\mathrm{d}m/\mathrm{d}z \neq m$. Thus, in (105) m is not the wavenumber unless m is constant. Nevertheless, this argument suggests that we seek solutions of a similar form to (105), and we find such solutions by way of a perturbation expansion below. We note that the condition that variations in m, or in the wavelength m^{-1}, occur only slowly may be expressed as

$$\frac{m}{|\partial m/\partial z|} \gg m^{-1} \quad \text{or} \qquad \left|\frac{\partial m}{\partial z}\right| \ll m^2. \tag{106}$$

This condition will generally be satisfied if variations in the background state, or in the medium, occur on a scale much longer than the wavelength.

B.1 Solution by perturbation expansion

To explicitly recognize the rapid variation of m we rescale the coordinate z with a small parameter ϵ; that is, we let $\widehat{z} = \epsilon z$ where $\widehat{z}$ varies by $\mathcal{O}(1)$ over the scale on which m varies. Eq. (104) becomes

$$\epsilon^2 \frac{\mathrm{d}^2\xi}{\mathrm{d}\widehat{z}^2} + m^2(\widehat{z})\xi = 0, \tag{107}$$

and we may now suppose that all variables are $\mathcal{O}(1)$. If m were constant the solution would be of the form $\xi = A\exp(m\widehat{z}/\epsilon)$ and this suggests that we look for a solution to (107) of the form

$$\xi(z) = \mathrm{e}^{g(\widehat{z})/\epsilon}, \tag{108}$$

where $g(\widehat{z})$ is some as yet unknown function. We then have, with primes denoting derivatives,

$$\xi' = \frac{1}{\epsilon} g' \mathrm{e}^{g/\epsilon}, \qquad \xi'' = \left(\frac{1}{\epsilon^2} g'^2 + \frac{1}{\epsilon} g''\right) \mathrm{e}^{g/\epsilon}. \tag{109a,b}$$

Using these expressions in (107) yields

$$\epsilon g'' + g'^2 + m^2 = 0, \tag{110}$$

and if we let $g = \int h \, \mathrm{d}\widehat{z}$ we obtain

$$\epsilon \frac{\mathrm{d}h}{\mathrm{d}\widehat{z}} + h^2 + m^2 = 0. \tag{111}$$

To obtain a solution of this equation we expand h in powers of the small parameter ϵ,

$$h(\widehat{z};\epsilon) = h_0(\widehat{z}) + \epsilon h_1(\widehat{z}) + \epsilon^2 h_2(\widehat{z}) + \cdots . \tag{112}$$

Substituting this in (111) and setting successive powers of ϵ to zero gives, at first and second order,

$$h_0^2 + m^2 = 0, \qquad 2h_0 h_1 + \frac{\mathrm{d}h_0}{\mathrm{d}\widehat{z}} = 0. \tag{113a,b}$$

The solutions of these equations are

$$h_0 = \pm \mathrm{i} m, \qquad h_1(\widehat{z}) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\widehat{z}} \ln \frac{m(\widehat{z})}{m_0}. \tag{114a,b}$$

where m_0 is a constant. Now, ignoring higher-order terms, (108) may be written in terms of h_0 and h_1 as

$$\xi(\widehat{z}) = \exp\left(\int h_0 \,\mathrm{d}\widehat{z}/\epsilon\right) \exp\left(\int h_1 \,\mathrm{d}\widehat{z}\right), \tag{115}$$

and, using (114) and with z in place of $\widehat{z}$, we obtain

$$\xi(z) = A_0 m^{-1/2} \exp\left(\pm \mathrm{i} \int m \,\mathrm{d}z\right). \tag{116}$$

where A_0 is a constant, and this is the WKB solution to (104). In general

$$\xi(z) = B_0 m^{-1/2} \exp\left(\mathrm{i} \int m \,\mathrm{d}z\right) + C_0 m^{-1/2} \exp\left(-\mathrm{i} \int m \,\mathrm{d}z\right). \tag{117}$$

or

$$\xi(z) = D_0 m^{-1/2} \cos\left(\int m \,\mathrm{d}z\right) + E_0 m^{-1/2} \sin\left(\int m \,\mathrm{d}z\right). \tag{118}$$

A property of (116) is that the derivative of the phase is just m; that is, m is indeed the local wavenumber. Note that a crucial aspect of the derivation is that m varies slowly, so that there is a small parameter, ϵ, in the problem. Having said this, it is often the case that WKB theory can provide qualitative guidance even when there is little scale separation between the variation of the background state and the wavelength. Asymptotics often works when it seemingly shouldn't.

References

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