Boundary Layers: Stratified Fluids

Lecture 3 by Jeroen Hazewinkel

continued from lecture 2

Using $w_I = E/(2\sigma S)\nabla^2 T_I$, the interior of the cylinder is be described by

$$\frac{E}{2\sigma S}\nabla^2 T_I + \frac{E^{1/2}}{4}\nabla^2 \int_0^1 T_I(r,z')dz' = \frac{E^{1/2}}{4}\frac{1}{r}(rv_T)_r. \tag{1}$$

This result can be rewritten as a Poisson equation

$$\nabla^2\Theta = \lambda\frac{1}{r}(rv_T)_r, \tag{2}$$

where we introduced the new temperature

$$\Theta = T_I + \lambda\int_0^1 T_I dz', \tag{3}$$

with

$$\lambda = \frac{\sigma S}{2E^{1/2}}. \tag{4}$$

This λ will be the determining parameter to the problem. Remember that, although being small, the term σS in λ is as yet undefined relatively to $E^{1/2}$. When λ is relatively small stratification is of little importance and the problem reduces to the unstratified Ekman problem. In the other limit, i.e. $\lambda \gg 1$, stratification suppresses most of the vertical motion.

In order to solve the Poisson equation 3 we need boundary conditions at all sides of the domain. At the top and bottom of the cylinder, we simply have to match the interior temperature with the forced exterior temperature. At the side walls we have to find solutions for $\Theta(r \to r_0)$. As we anticipate side wall boundary layers, we introduce a boundary layer correction to all variables, e.g. $u = u_I(r) + u_{bl}(\zeta) \to u_{bl} = 0, \zeta \to \infty$. As before we will use a boundary layer coordinate ζ. Note that the boundary layer scale has to be determined as yet, i.e. $r = \delta\zeta$ with δ unknown. In order to find this scale we turn back to the same balance equations that hold in the interior. However, the full Laplacian is replaced by its first approximation in the rapidly changing coordinate ζ, i.e. $\nabla \to \partial_{\zeta\zeta}$. This gives

$$v = p_\zeta, \tag{5}$$

$$u = \frac{E}{2} v_{\zeta\zeta}, \tag{6}$$

$$0 = p_z + T + \frac{E}{2} w_{\zeta\zeta}, \tag{7}$$

$$u_\zeta + w_z = 0, \tag{8}$$

$$\sigma S w = \frac{E}{2} T_{\zeta\zeta}. \tag{9}$$

Combination of the above equations results in $w_z = -E/2p_{4\zeta}$ and $T_z = -\sigma S p_{\zeta\zeta}$. Taking the z-derivative of 7 and using the above found relations gives one equation for pressure p,

$$\frac{E^2}{4} p_{6\zeta} + \sigma S p_{\zeta\zeta} + p_{zz} = 0. \tag{10}$$

Recall that we assume all derivatives to z are $O(1)$ and those to ζ are $O(1/\delta)$. This means that for example the first term of 10 is $O(E^2/\delta^6)$.

There are several balances that satisfy 10. We will examine the options. Firstly, there could be a balance between the first term and the last term. This implies that $E^2/\delta^6 = O(1), -> \delta = E^{1/3}$, the so called Stewardson layer. This layer exists for a homogeneous, rotating fluid. In order for the second term of 10 to be negligible we also find that $\sigma S \ll E^{2/3}$.

Considering $\sigma S \gg E^{2/3}$ there are two possible balances. For $\delta = E^{1/2}/(\sigma S)^{1/4}$ the first and second term balance. This boundary layer only depends on the stratification and is therefore called the buoyancy layer. For $\delta = (\sigma S)^{1/2}$ we find a balance between the second and third term of 10. This layer is the hydrostatic layer. As both boundary and hydrostatic layers are found in the limit $\sigma S \gg E^{2/3}$ they coexist. In the larger hydrostatic layer, close to the side wall the buoyancy layer is found. With decreasing stratification these two combine in the Stewardson layer. In both limits, σS small and large compared to $E^{2/3}$, the full sixth order of 10 is preserved. As we considered the cylinder with a stratification we will have to see the impact of both hydrostatic and buoyancy layers on the interior, or how they set the boundary condition for 3.

0.1 Hydrostatic boundary layer

Turning to the hydrostatic layer we introduce a stretched coordinate

$$\eta = \frac{r_0 - r}{\delta_{hydrostatic}} = \frac{r_0 - r}{(\sigma S)^{1/2}}. \tag{11}$$

We rewrite the azimuthal velocity as $v = \widetilde{v}(r, \eta)$. This indicates, from the governing equations, that $u = E/(\sigma S)\tilde{u}$ and $p = (\sigma S)^{1/2}$. The governing equations, accurate to terms of order larger than $E^{2/3}$, become

$$\widetilde{v} = \widetilde{p}_\eta,$$

$$\widetilde{u} = \frac{1}{2}\widetilde{v}_{\eta\eta}, \tag{12}$$

$$\widetilde{T} = \widetilde{p}_z, \tag{13}$$

$$\widetilde{u}_\eta + \widetilde{w}_z = 0, \tag{14}$$

$$\widetilde{w} = \frac{1}{2}\widetilde{T}_{\eta\eta}. \tag{15}$$

Friction is in this layer not of importance, as we balanced the second and third term in 10. Combining all terms we find an equation for v. These can be combined to give, (with the assumption) that v vanishes at $r = r_0$

$$\widetilde{v}_{\eta\eta} + \widetilde{v}_{zz} = 0. \tag{16}$$

To solve this we notice that the thickness of the hydrostatic layer, in the case of $\sigma S \gg E$, is much greater than that of the Ekman layer at the top and bottom. This means that for these Ekman layers the dynamics of the hydrostatic layer is part of the interior. The vertical velocity shows that

$$w_h = \frac{E}{(\sigma)^{3/2}}\widetilde{w} = \frac{E}{(\sigma)^{3/2}}\widetilde{T}_{\eta\eta} = \frac{E}{(\sigma)^{3/2}}\widetilde{v}_{z\eta}. \tag{17}$$

From the Ekman transport we find that this vertical velocity is

$$\frac{E}{2}\widetilde{v}_{hr} = \frac{E^{1/2}}{(\sigma)^{1/2}}\widetilde{v}_\eta, \tag{18}$$

indicating that $\widetilde{v}_{z\eta} = \sigma S/E^{1/2}v_\eta$ or $\widetilde{v}_z = \lambda\widetilde{v}$ at top and bottom of the hydrostatic layer. Complimentary to this we will assume that $\tilde{v} = V(z)e^{-a\eta}$, so that 16 results in

$$(\lambda^2 + a^2)\tilde{v} = 0. \tag{19}$$

This shows that the hydrostatic boundary layer has a characteristic scale of $a = (2\sigma S)^{1/2}/E^{1/4}$.

0.2 Buoyancy layer

In the very thin buoyancy layer we found the thickness to be $\delta_b = E^{1/2}/(\sigma S)^{1/4}$. Introducing the boundary layer coordinate $\xi = (r - r_0)/\delta_b$ and $v = \hat{v}$ etc., we rewrite the governing equations again to find

$$\hat{v} = \hat{p}_\xi, \tag{20}$$

$$\hat{u} = \frac{1}{2}\hat{v}_{\xi\xi}, \tag{21}$$

$$\hat{T} = \frac{1}{2}\hat{w}_{\xi\xi}, \tag{22}$$

$$\hat{u}_\xi + \hat{w}_z = 0, \tag{23}$$

$$\hat{w} = \frac{1}{2}\hat{T}_{\xi\xi}. \tag{24}$$

Note that, as we derived the buoyancy layer being the balance between buoyancy and the friction we also see this in the third equation above. We see that combining the equations implies that

$$\hat{T}_{4\xi} + 4\hat{T} = 0, \tag{25}$$

indicating an Ekman like boundary layer. Using the Ekman solution we find

$$\hat{T} = Ae^{-\xi}\cos\xi + Be^{-\xi}\sin\xi. \tag{26}$$

As $\hat{\psi} = \hat{T}_\xi$ we find that

$$\hat{\psi} = \frac{r_0}{2}[-Ae^{-\xi}(cos\xi + sin\xi) + Be^{-\xi}(cos\xi + sin\xi)]. \tag{27}$$

Now we have expressions for both hydrostatic and buoyancy layers. The sum of the solutions has to meet the outer boundary conditions.

0.3 Matching at $r = r_0$

At the outer rim, $r = r_0$, there is no slip which means that

$$v_I(r_0, z) + \tilde{v}(0, z) + \frac{E}{(\sigma S)^{3/2}}\hat{v}(0, z) = 0. \tag{28}$$

The last term on the lhs is negligible so we find that interior velocity is balanced by the velocity from the hydrostatic layer. Taking the z-derivative and using the thermal wind relation, **??**, leads to

$$(T_I)_r - \tilde{T}_\eta = 0. \tag{29}$$

In case of an insulating side wall at $r = r_0$ the radial derivative of total temperature should be zero, i.e. $(T_I)_r - \tilde{T}_\eta - T_\xi = 0$. Combining these two results we see that $T_\xi = 0$. Also the total stream function should be zero. Combining previously found expressions for ψ we find

$$0 = \psi_I + \frac{E}{(\sigma S)}\hat{\psi} + \frac{E}{\sigma S}\tilde{\psi} = \psi_I - \frac{Er_0}{2\sigma S}\tilde{T}_\eta - \frac{Er_0}{2\sigma S}\hat{T}_\xi. \tag{30}$$

Noting that we found an expression for $\tilde{T}(r_0)_\eta$ and $\hat{T}(r_0)_\xi = 0$, we see that

$$\psi_I = \frac{Er_0}{2\sigma S}(T_I)_r. \tag{31}$$

By definition $\psi_r = rw$ and we recall the interior balance between vertical flux and temperature and forcing velocity. We recast this at position $r = r_0$ so that we can use 31, and find

$$\frac{E^{1/2}}{4}r_0 v_T - \frac{E^{1/2}}{4}r_0\int_0^1 \partial_r T_I dz - \frac{r_0 E}{2\sigma S}r_0\int_0^1 \partial_r T_I dz = 0. \tag{32}$$

Assuming that there is no z dependence for the interior temperature, the integrals give us the integrants so we have

$$\partial_r T_I = \frac{\lambda}{1+\lambda}v_T(r_0). \tag{33}$$

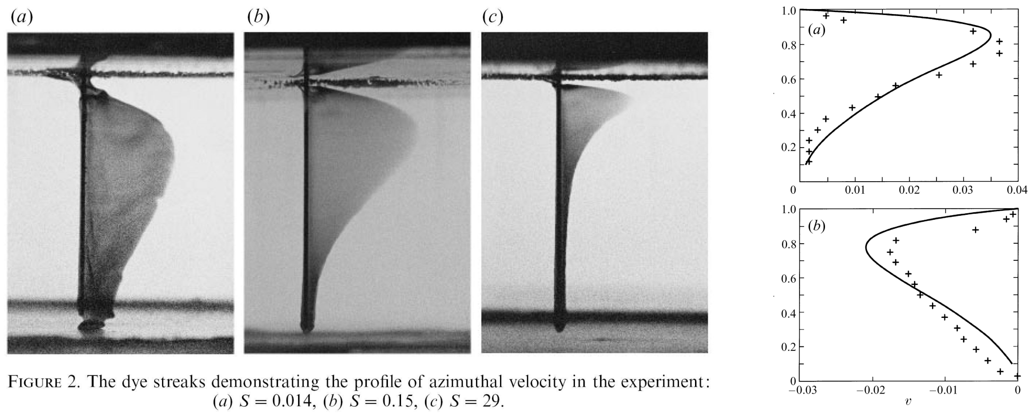

Figure 1: Observations by Pedlosky et al. (1997) in a rotating cylinder with a temperature profile on top. Shown are three different experiments in which the azimuthal velocity is visualized by dye. Note that the profiles changes for different stratification. On the right their comparison between observations (+) and theoretical predicted profile (line)

The z independence of T also means that $\Theta(r_0)_r \equiv T_r(r_0)+\lambda \int_0^1 T_r(r_0)dz = (1+\lambda)T_r(r_0) = \lambda v_T(r_0)$. This means that we finally found the boundary condition for 3 being

$$\Theta_r(r_0) = \lambda v_t(r_0). \tag{34}$$

1 Two experiments

We will briefly discuss two experiments that can be seen as confirmation of the above theory for a rotating, stratified fluid. In their experiment **?** considered a cylinder, having a temperature on top in the varying in the radial direction and bottom at a fixed temperature. The boundary on top did not rotate differentially from the rotation of the whole tank. In this special set-up it turns out that on the side walls both hydrostatic and buoyancy layers are inactive but a layer of the scale $(\sigma S)^{1/2}$ exists in the vicinity of the upper boundary. For sufficiently large stratification, the Ekman pumping into the interior is completely suppressed. The nearly inviscid interior velocity field has to match the no-slip in a layer that decreases with decreasing stratification. Beneath this layer, a smooth transition of the azimuthal velocity towards null at the bottom is found. In a comparison between theory and experiments the velocity profiles were found to be in good agreement, as shown in Figure 1.

In a second study Whitehead and Pedlosky (2000), again a stratified fluid in a rotating cylinder was considered. However, in this case not the top, but a coil around the cylinder heated the fluid. The heating was placed in the middle of the height of the cylinder. This sidewall warming forces a vertical mass flux in the sidewall boundary layer. The divergence of this flux effects the interior flow and drives a azimuthal velocity. Again, the theoretical predictions and observations were in close agreement. In Figure 2 both theory and experimental velocity profiles at several radii are shown. As the stratification increased, Whitehead and Pedlosky (2000) observed that the velocity profile got a sharper peak.

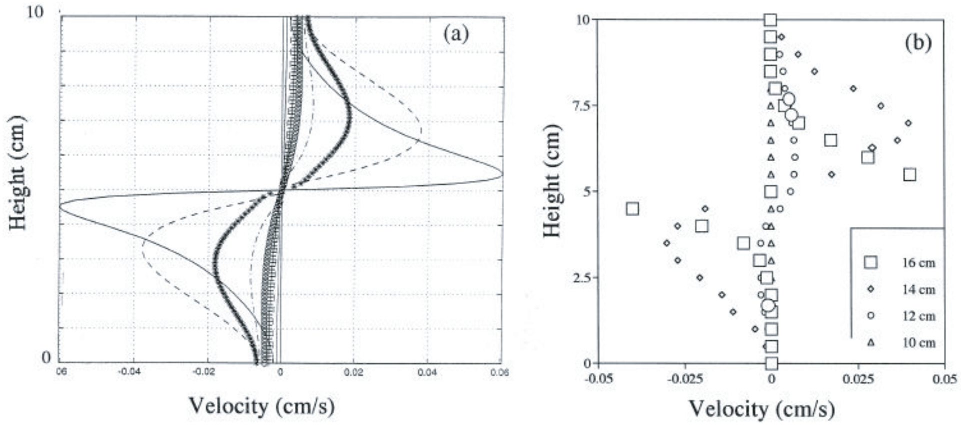

Figure 2: Comparison between a) theoretical prediction for the velocity profiles and b) the observations. The various lines and dots correspond to different sampling radii. in Whitehead and Pedlosky (2000).

References

J. Pedlosky, J. A. Whitehead, and G. Veitch. Thermally driven motions in a rotating stratified fluid. theory and experiment. *Journal of Fluid Mechanics*, 339:391–411, 1997.

J. A. Whitehead and J. Pedlosky. Circulation and boundary layers in differentially heated rotating stratified fluid. *Dynamics of Atmosphere and Oceans*, 31:1–21, 2000.