Boundary Layers

Lecture 2 by Basile Gallet

continued from lecture 1

This leads to a differential equation in Z

$$\frac{\partial}{\partial Z}(A - iB) + (A - iB)[-C + \frac{U_y}{2i(1+i)}] = 0 \text{ with } C = -\frac{U_y}{2}.$$

The solution to this equation with the boundary conditions $A(0) = U$ and $B(0) = 0$ is

$$A(Z) = Ue^{-U_y Z/4} cos(U_y Z/4) \quad (1)$$

$$B(Z) = -Ue^{-U_y Z/4} sin(U_y Z/4). \quad (2)$$

Hence the corrected velocity field is

$$u_0 = U[1 - e^{-\zeta(1+\epsilon U_y/4)} cos(\zeta(1 - \epsilon U_y/4))] \quad (3)$$

$$v_0 = U[e^{-\zeta(1+\epsilon U_y/4)} sin(\zeta(1 - \epsilon U_y/4))]. \quad (4)$$

If $U_y < 0$, ω_z is positive, but the boundary layer is bigger than without the nonlinear correction : sacrebleu! Our guess on the size of the boundary layer $\delta = \sqrt{\frac{2\nu}{f+\omega_z}}$ was wrong. The main nonlinear effect is not that local vorticity should be added to the global rotation. There is a stronger effect which is that the vorticity in the boundary layer is advected in the z direction by the pumping velocity w ($w > 0$ for $U_y < 0$).

The equation for Λ_1 can then be solved to get the first order correction to u and v, and finally to w using the mass conservation equation

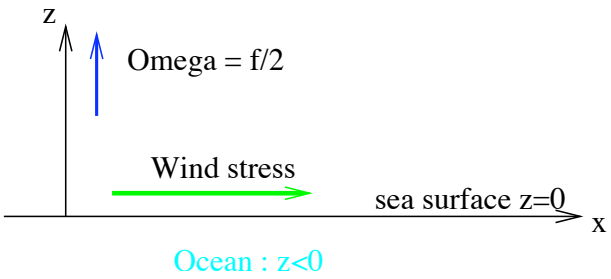
$$w_1 = -E^{1/2}[\frac{1}{2}U_y + \frac{7\epsilon}{40}(U_y^2 + UU_{yy})]. \quad (5)$$

The effects of the nonlinear terms on the pumping depend on the structure of the function $U(y)$, e.g. Uyy.

Boundary Layers: Stratified Fluids

1 Nansen's problem

1.1 Effect of the wind on the oceanic currents



The problem in which the wind is applying a constant stress $\tau(y)$ in the x direction to the surface of the ocean is known as Nansen's problem. If the ocean is at rest, the velocity vanishes at great depth. The stress applied by the wind will induce a current in a boundary layer near the surface. If the Rossby number ϵ is small and the flow is steady, the Coriolis force will counterbalance the stress imposed by the wind. Since the velocity is significantly non-zero only inside the boundary layer, one can get an order of magnitude, U_0, of the velocity of the current from the equilibrium of these two terms

$$\tau \sim 2\rho\Omega U_0 \delta_e \text{ which gives } U_0 = \frac{2\tau}{\rho f \delta_e} \text{ with } f = 2\Omega.$$

The velocities can be rescaled by U_0, the horizontal coordinates by L and the altitude by the height of the boundary layer δ_e. This leads to the variable $\zeta = \frac{z}{L}E^{-1/2}$. If we consider $\epsilon << 1$ we can rewrite the Navier-Stokes equation in terms of these non-dimensional variables

$$\begin{aligned} -v &= -p_x + u_{\zeta\zeta} && (6)\\ u &= -p_y + v_{\zeta\zeta} && (7)\\ 0 &= -p_\zeta. && (8) \end{aligned}$$

The pressure is independent of z (we included the gravity term in it) and is equal to its value outside the ocean, which does not depend on x and y. The system of equations reduces to

$$\begin{aligned} -v &= u_{\zeta\zeta} && (9)\\ u &= v_{\zeta\zeta}. && (10) \end{aligned}$$

This can be written in terms of the complex variable $\Lambda = u + iv$ and leads to

$$\Lambda\zeta\zeta - i\Lambda = 0. \quad (11)$$

This problem is the same as the Ekman layer problem except for the boundary conditions on the velocity, which are $v_z = 0$ and $u_z = \tau$ at $z = 0$ and $u = v = 0$ at $z = -\inf$. The velocity field that satisfies these boundary conditions is

$$u = \frac{\tau}{\sqrt{(2)}} e^{\zeta} cos(\zeta - \frac{\pi}{4}) \tag{12}$$

$$v = \frac{\tau}{\sqrt{(2)}} e^{\zeta} sin(\zeta - \frac{\pi}{4}). \tag{13}$$

One should note that the velocity at the surface makes a 45^o angle on the right of the surface stress vector (for $\Omega > 0$) : the current induced by the wind does not drive the objects floating in the ocean exactly in the direction of the wind. They are deviated to the right in the North hemisphere and to the left in the South hemisphere. The hodograph shows that the velocities are very close to zero as soon as ζ is under -1. The vertical velocity is rescaled like the altitude and leads to the new variable $W = E^{-1/2} w$ which can be computed from the mass conservation equation

$$v_y + W_\zeta = 0 \text{ gives } W = -\tfrac{1}{2}\tau_y (1 - e^{\zeta} cos(\zeta)).$$

It has a non-zero limit at great depths, so

$$W(-\inf) = -\frac{1}{2}\tau_y. \tag{14}$$

For a wind stress of $1 dyne/cm^2$ this is a velocity of $10 cm/day$. This phenomenon is responsible for a major part of the ocean circulation.

1.2 Role of non-linearities

We can now try to figure out what the effects of the nonlinear terms are. We consider the situation in which the velocity in the x direction is not zero in the deep ocean, but has a finite limit $u_g(y)$.

If the Rossby number is smaller than one but "not so small", one may want to calculate the effect of non-linearities on the solution we found, which requires a new depth scale and a new variable $Z = \epsilon\zeta$. The velocity components are developed in powers of ϵ : $u = u_0 + \epsilon u_1, v = v_0 + \epsilon v_1$. As we did in the previous section, the zero order equation gives the solution we have just calculated but with constants of integration which depend on the variable Z. The first order equation in the equation of an oscillator with a "forcing" on the right hand side. For the development to be consistent there must not be any component of this forcing at the resonant frequency of the oscillator. This condition leads to a differential equation in Z for the integration constants of the zero order solution. We can solve this equation and get the corrected velocity

$$u_0 = u_g + \frac{\tau}{\sqrt{(2)}} e^{\zeta(1-\epsilon(\frac{\tau_y}{2} + \frac{u_{gy}}{4}))} cos(\zeta(1 - \epsilon u_{gy}/4) - \frac{\pi}{4}) \tag{15}$$

$$v_0 = \frac{\tau}{\sqrt{(2)}} e^{\zeta(1-\epsilon(\frac{\tau_y}{2} + \frac{u_{gy}}{4}))} sin(\zeta(1 - \epsilon u_{gy}/4) - \frac{\pi}{4}). \tag{16}$$

The boundary layer thickness in dimensional form is

$$\delta_* = \sqrt{\frac{2\nu}{f - \frac{2\tau_{*y}}{\rho f \delta} - \frac{u_{*gy}}{2}}}. \tag{17}$$

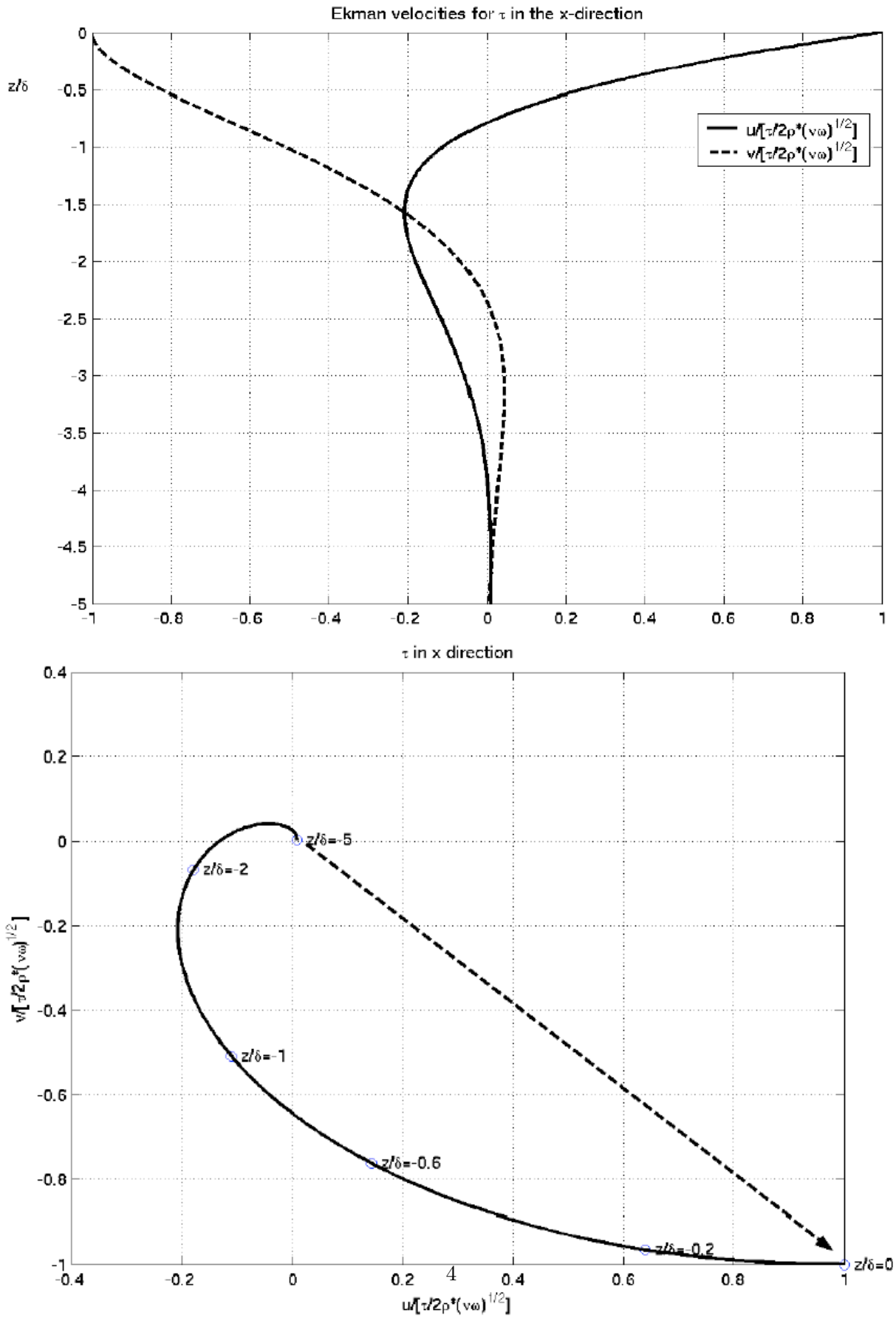


Figure 1: Top : Horizontal components of the Velocity field as a function of z/δ. Bottom : Hodograph of the horizontal components of the velocity field.

We see that in this case the boundary layer is thinner if $u_{gy} < 0$. It's not u_{gy} which create the vertical flow but the curl of the stress $-\tau_y$. The vertical advection is then independent of u_{gy} and the only effect of the vertical vorticity is to add to the global rotation to make a thinner boundary layer.

We can also compute the order one correction to the velocity field and the correction to the vertical flow

$$W(-\infty) = -\frac{1}{2}\frac{\partial}{\partial y}[\frac{\tau}{1-\epsilon(u_{gy}+\frac{\tau_y}{4})}]. \tag{18}$$

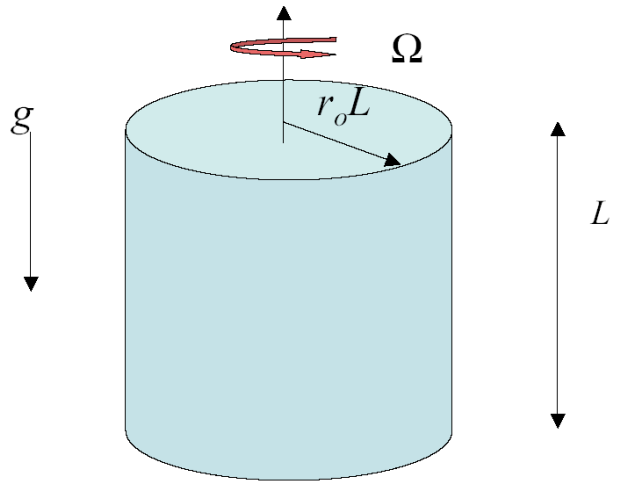
This vertical flow depends on the horizontal velocity gradients in the deep ocean only at the first order.

One may wonder if a linearization of the nonlinear terms around u_e is still possible in the case $u_g >> u_e$. For more information on this subject one can look at Thomas and Rhines (2002).

Stratified Fluids

In an ocean there are changes in the density of the water due to variations both in the salinity and the temperature of the water. The presence of density stratification introduces new and very interesting elements to the boundary layer picture and the control of the interior flow by the boundary layers. The control of the boundary layer on the interior flow is mediated by vertical inertial waves. The stratification allows information to propagate horizontally through internal gravity waves. There will be some kind of competition between the top and bottom boundary conditions on the one hand, which propagate through inertial waves, and the side walls boundary conditions, which propagate through internal gravity waves.

2 The cylinder problem



An interesting problem is to study the motion of a stratified fluid inside a rotating cylinder. The cylinder's axis is vertical. The height of the cylinder is L and its radius is $r_0 L$. We can use the following scalings:

$$\overrightarrow{u_*} = U\overrightarrow{u}, \tag{19}$$

$$\overrightarrow{x_*} = L\overrightarrow{x}, \tag{20}$$

$$T_* = \Delta T_v(z_*/L) + \Delta T_h T(x,y,z). \tag{21}$$

In these equations, the variables with a star are dimensional. ΔT_v and ΔT_h are the vertical and horizontal temperature variations over the size of the cylinder. The density is supposed to be linear in the temperature variations

$$\rho_* = \rho_0[1 - \alpha T_*]. \tag{22}$$

We define p as the non-dimensional pressure difference to the hydrostatic pressure

$$p_* = \rho_0 g\alpha\Delta T_v(z_*^2/(2L)) + \rho_0 fULp(x,y,z), \text{ with } f = 2\Omega.$$

The scaling velocity is given by a thermal wind balance : $U = \frac{\alpha g \Delta T_h}{f}$. The motion is supposed to be steady, incompressible and to follow the Boussinesq approximation. The equations are :

- Navier-Stokes $\epsilon(\overrightarrow{u}.\overrightarrow{\nabla})\overrightarrow{u} + \overrightarrow{k}\,\Lambda\,\overrightarrow{u} = -\overrightarrow{\nabla}(p) + T\overrightarrow{k} + \frac{E}{2}\nabla^2\overrightarrow{u}$
- Incompressibility $\overrightarrow{\nabla}.\overrightarrow{u} = 0$
- Temperature advection-diffusion $\epsilon(\overrightarrow{u}.\overrightarrow{\nabla})T + wS = \frac{E}{2\sigma}\nabla^2 T$

These 3 equations involve 4 dimensionless numbers :

- The Rossby number $\epsilon = \frac{U}{fL}$
- The Ekman number $E = \frac{2\nu}{fL^2}$
- The Prandtl number $\sigma = \frac{\nu}{\kappa}$, where κ is the thermal diffusivity of the fluid.
- A characteristic number of the vertical stratification $S = \frac{\alpha g \Delta T_v}{f^2 L} = \frac{N^2}{f^2}$, where N is the oscillation frequency of inertial waves inside the cylinder.

We consider that the ratio $\frac{\epsilon}{S} = \frac{\Delta T_h}{\Delta T_v}$ is small, so that the temperature equation can be linearized. We can impose different boundary conditions to the flow :

- We assume a no-slip boundary condition. There must also be no flow perpendicular to the boundary.
- One or more boundaries may be moving in their own plane.
- The cylinder walls may either be insulating or at a fixed given temperature.

We use polar coordinates, with u being the radial velocity, v the azimuthal one, and w the vertical one. We assume that the Rossby number is small so that the equations of motion are linear. For an axially symmetric motion we get

$$-v = -p_r + \frac{E}{2}[\nabla^2 u - \frac{u}{r^2}] \quad (23)$$

$$u = \frac{E}{2}[\nabla^2 v - \frac{v}{r^2}] \quad (24)$$

$$0 = -p_z + T + \frac{E}{2}\nabla^2 w \quad (25)$$

$$\frac{1}{r}(ru)_r + w_z = 0 \quad (26)$$

$$w\sigma S = \frac{E}{2}\nabla^2 T. \quad (27)$$

We see in the last equation that the perturbation due to vertical motion is balanced in the steady state by diffusion. The interior vertical velocity "tolerated" by the system is $w_I \sim \frac{E}{\sigma S}$. However, an Ekman boundary layer induces a vertical pumping which velocity is of order $E^{1/2}$. This means that for a low Ekman number the vertical pumping can be much stronger than the vertical velocity tolerated by the interior. We may wonder how the system is going to respond to such a perturbation. The Ekman layers are found using $z = \zeta E^{1/2}$ and $w = WE^{1/2}$

$$
\begin{aligned}
-v &= -p_r + \frac{1}{2}u_{\zeta\zeta} && (28)\\
u &= \frac{1}{2}v_{\zeta\zeta} && (29)\\
0 &= -p_\zeta + E^{1/2}T + \frac{1}{2}EW_{\zeta\zeta} && (30)\\
(ru)_r + rW_\zeta &= 0 && (31)\\
\sigma S E^{1/2} W &= \frac{1}{2}T_{\zeta\zeta}. && (32)
\end{aligned}
$$

We see in the last equation that if $\sigma S << E^{-1/2}$ we can ignore the buoyancy forces in the Ekman layer. The temperature is the same as in a purely diffusive state. The boundary layer is so thin that it remains unchanged. One should remember however that this is true only for a horizontal bottom boundary.

We can adapt the previous results for the Ekman layer to polar coordinates and get the compatibility condition between the boundary layer flow and the flow in the interior of the cylinder (variables with a subscript I)

$$\int_0^{+\infty} u(\zeta)d\zeta = \frac{v_I}{2} \text{ and } w_I(r,0) = E^{1/2}W(r,+\infty)$$

.

An integration with respect to ζ of the mass conservation equation leads to

$$w_I(r,0) = E^{1/2}W(r,+\infty) = \frac{E^{1/2}}{2r}(rv_I)_r. \quad (33)$$

If the upper boundary is rotating with differential speed $v_T(r)$, the same analysis yields

$$w_I(r,1) = \frac{E^{1/2}}{2r}\frac{\partial}{\partial r}(r(v_T - v_I(r,1))). \quad (34)$$

In the limit $E << 1$ the equations governing the interior flow are

$$
\begin{aligned}
v_I &= p_{Ir} && (35)\\
T_I &= -p_{Iz} && (36)\\
u_I &= \frac{E}{2}[\nabla^2 v - \frac{v}{r^2}] && (37)\\
\frac{1}{r}(ru_I)_r + w_{Iz} &= 0 && (38)\\
w_I &= \frac{E}{2\sigma S}\nabla^2 T_I. && (39)
\end{aligned}
$$

u_I is of order E, which means that if $\sigma S << 1$ then $w_I >> u_I$, so that the mass conservation equation becomes $\frac{\partial}{\partial z} w_I = 0$. The interior vertical velocity is independent of z and may be written as the mean of its values at $z = 0$ and $z = 1$

$$w_I = \frac{1}{2}(w_I(r,1) + w_I(r,0)) = \frac{E^{1/2}}{4r}\frac{\partial}{\partial r}[r(v_T - (v_I(r,1) - v_I(r,0)))]. \tag{40}$$

If we differentiate the azimuthal component of the Navier-Stokes equation in the interior with z and its vertical part with r we get the thermal wind equation

$$\frac{\partial}{\partial z} v_I = \frac{\partial}{\partial r} T_I. \tag{41}$$

This equation can be integrated with respect to z

$$v_I(r,1) - v_I(r,0) = \frac{\partial}{\partial r}\int_0^1 T_I(r,z')dz'. \tag{42}$$

The vertical velocity of the interior can then be written in terms of the temperature and of the forcing velocity

$$w_I = \frac{E^{1/2}}{4r}\frac{\partial}{\partial r}(r v_T) - \frac{E^{1/2}}{4r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}(\int_0^1 T_I(r,z')dz')). \tag{43}$$

References

L. N. Thomas and Peter B. Rhines. Nonlinear stratified spin-up. *Journal of Fluid Mechanics*, (473):211–244, 2002.