Lecture 8: The El Niño Phenomenon (continued)

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1 Deterministic Chaos

So far we have seen that the oscillatory behavior of El Nino Southern Oscillation (ENSO) mechanism was related to the saturation of a unstable mode above the threshold of a Hopf bifurcation, corresponding to a given critical coupling strength between the atmosphere and the ocean. An interesting point in the record of the ENSO signal is the signature of some mean seasonal cycle. For the western tropical Pacific ocean, negative anomalies in the records of zonal winds occur around April, whereas positive anomalies occur around December. Furthermore, sea surface temperature (SST) anomalies are observed at the same periods: positive SST anomalies are associated with negative zonal wind anomalies, and negative SST anomalies are associated with positive zonal wind anomalies. It is important to note that ENSO events and the seasonal cycle are sensitive to the same environmental factors such as wind forcing and the ocean circulation. ENSO's non-linear interaction with the seasonal cycle is characterized by a tendency to synchrony in periodic, subharmonic oscillation. At this point we saw an interesting movie: illustration of the 5 unsynchronized oscillators metronoms. Once they are coupled together through a moving plate, their oscillation period tend to synchronize. Once the plate is removed (the coupling is broken), the periods desynchronize again.

In order to illustrate the tendency for phase-locking of anomalous events, we pick up a constant driving frequency Ω and a starting point x_0, then iteratively compute x_{n+1} from x_n through the non-linear relationship:

$$x_{n+1} = x_n + \Omega - \frac{k}{2\pi} \sin\big(2 * \pi x (mod 1)\big) \tag{1}$$

The map of the interaction between anomalies and the driving cycle is obtained by displaying all the computed x in the (Ω, x) plane, for all the described Ω. As we increase the parameter k in the non-linear forcing, an increasing number of "windows" are opening and widening in the (Ω, x) space. These "windows" are the orbital periods of limit circles encountered in the iterative process, and correspond to rational multiples of the driving frequency Ω onto which the system is locking. The frequency ratio of the model to the driving frequency describe a "devil's staircase" as the number and width of frequency-locked steps (corresponding to the windows) increases infinitely.

The Zebiak-Cane model is the first coupled atmosphere-ocean model taking into account the interaction between the seasonal cycle and the Hopf bifurcation oscillatory frequency by relating the seasonal frequency Ω to the atmospheric/oceanic coupling strength parameter μ and the upwelling feedback parameter δ through a "Devil's terrace" (we recover a Devil's staircase with two non-linear forcing terms, equivalent of a k_1 and a k_2 in previous description). As the coupling strength is increased, both the amplitude and the time scale of the oscillations are enhanced. By tuning the parameters μ and δ, Zebiak-Cane model predicts 3 ENSO events over 10 years, which is in good agreement with observational data.

This ends the deterministic part of the discussion. Stochasticity will be now included by adding some noise into ENSO models.

2 Effect of Noise on the Hopf Bifurcation

Unresolved fast and short scales can be integrated into ENSO models by adding noise. As an example, the westerly wind bursts (WWB) events are characterized by velocities above 7 m s^{-1}, with a typical duration of a few days. These unresolved processes are known to trigger the propagation of perturbations in the form of equatorial Kelvin and Rossby waves. The correlation between this events can be verified in using a singular value decomposition analysis of the SST-Wind covariance matrix.

If we assume that the effect of WWB is a noise in the system, then what is the response of the model ?

The response of Zebiak-Cane model to white versus red noise is represented in the subcritical and supercritical regimes. A remarkable result is that red noise can trigger a response even before the critical point for the Hopf bifurcation is reached, that is, while still in the subcritical regime. Adding red noise in the model can thus result in lowering the bifurcation threshold.

We consider the following normal form:

$$\dot{X} = \lambda X - \omega Y - X(X^2 + Y^2), \tag{2}$$

$$\dot{Y} = \lambda Y + \omega X - Y(X^2 + Y^2). \tag{3}$$

In our model, X would be the temperature anomaly on the East Pacific coast (at some fixed longitude, say 30E) and Y the thermocline depth on the West coast. We derive the stochastic extension of this normal form by adding some noise in the equation (where dW_1 and dW_2 are independent, Gaussian noises).

$$dX = (\lambda X - \omega Y - X(X^2 + Y^2))dt + \sigma dW_1, \tag{4}$$

$$dY = (\lambda Y + \omega X - Y(X^2 + Y^2))dt + \sigma dW_2. \tag{5}$$

In polar coordinates ($r = \sqrt{X^2 + Y^2}$ and $\theta = \arctan \frac{X}{Y}$), the system reads (using Ito's formula for change of variables):

$$
\begin{aligned}
dR =& \frac{\partial r}{\partial x}dX + \frac{\partial r}{\partial y}dY + \frac{1}{2}\Big(\frac{\partial^2 r}{\partial x^2}(dX)^2 + \frac{\partial^2 r}{\partial y^2}(dY)^2\Big) + ...\\
=& \frac{X}{r}\sigma dW_1 + \frac{Y}{r}\sigma dW_2 + \frac{1}{2}\Big(\frac{Y^2}{r^3}(\sigma)^2 + \frac{X^2}{r^3}(\sigma)^2\Big)dt + ...\\
=& \underbrace{\sigma(\cos\theta dW_1 + \sin\theta dW_2)}_{\text{noise term}} + \underbrace{\frac{\sigma^2}{2r}dt}_{\text{additional drift term}} ,\\
d\theta =& -\frac{y}{r^2}\sigma dW_1 + \frac{x}{r^2}\sigma dW_2 + \frac{1}{2}\Big(\frac{2XY}{r^4}(\sigma)^2 - \frac{2XY}{r^4}(\sigma)^2\Big)dt + ...\\
=& \underbrace{\sigma\Big(\frac{\cos\theta}{r}dW_2 - \frac{\sin\theta}{r}dW_1\Big)}_{\text{additional drift term}} .
\end{aligned}
$$

We now make the following transformation

$$
\begin{aligned}
dX &= \sigma dW_1,\\
dY &= \sigma dW_2.
\end{aligned}
$$

$$
dR = (\lambda r - r^3 + \frac{\sigma^2}{2r})dt + \sigma(\cos\theta dW_1 + \sin\theta dW_2), \tag{6}
$$

$$
d\theta = \sigma(\frac{\cos\theta}{r}dW_2 - \frac{\sin\theta}{r}dW_1) + \omega dt. \tag{7}
$$

The stationary probability density function is

$$
\rho_S(r) = N\exp\Big(\frac{\lambda r^2}{\sigma^2} - \frac{r^4}{2\sigma^2}\Big), \tag{8}
$$

To prove (8) we first derive a system of coupled Fokker-Planck equations associated with the stochastic system above. We obtain:

$$
\underline{f} = \begin{pmatrix} \lambda r - r^3 + \frac{\sigma^2}{2r} \\ \omega \end{pmatrix} \tag{9}
$$

$$
\underline{\underline{g}} = \sigma \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix} \tag{10}
$$

The diffusion operator D_{ij} is then

$$
\underline{\underline{D}} = \underline{\underline{g}}\underline{\underline{g}}^T = \sigma^2 \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix} \tag{11}
$$

We look for a stationary density function $\rho_S(r)$, that is, $\partial_t \rho_S(r) = 0$). Thus

$$
0 = -\frac{1}{r}\Big((\lambda r - r^3 + \frac{\sigma^2}{2r})r\rho^S\Big)' + \frac{\sigma^2}{2r}(r\rho_S')
$$

$$\frac{\rho'^S}{\rho^S} = -\frac{2}{\sigma^2}\lambda r - 2\frac{r^3}{\sigma^2} + \frac{1}{r}$$
$$\ln \rho^S = -\frac{\lambda}{\sigma^2}r^2 - \frac{r^4}{2\sigma^2} + \ln r$$

Finally

$$\rho^S(r) = N \exp\big(\frac{\lambda r^2}{\sigma^2} - \frac{r^4}{2\sigma^2}\big) \tag{12}$$

3 Stochastic Optimals

What forcing pattern is maximizing the variability of the system under subcritical conditions? In order to address this question, we introduce white noise (delta-correlated noise in time) with a spatial correlation given by the covariance matrix C:

$$\frac{d\Psi}{dt} = A(t)\Psi + f(t) \tag{13}$$

$$\Psi_n = \Psi_{n-1} + A_{n-1}\Psi_{n-1}dt + \underbrace{\sqrt{dt}\zeta_{n-1}}_{\text{forcing}} \tag{14}$$

$$\mathbb{E}(\zeta_i) = 0, \tag{15}$$
$$\mathbb{E}(\zeta_i\zeta_j) = \delta_{ij}C \tag{16}$$

Now rewrite:

$$\begin{aligned}\Psi_n =&(1 + A_{n-1}dt)\Psi_{n-1} + \sqrt{dt}\zeta_{n-1}\\ =&(1 + A_{n-1}dt)\big((1 + A_{n-2}dt)\Psi_{n-2} + \sqrt{dt}\zeta_{n-1}\big) + \sqrt{dt}\zeta_{n-2}\\ =&...\end{aligned}$$

By recurrence, we find:

$$= R_{0,n}\Psi_0 + \sqrt(dt)\sum_{k=0}^{n-2} R_{k+1,n}\zeta_k, \tag{17}$$

where we have introduced the "propagator" $R_{n-1,n}$ such that

$$\Psi_n = (1 + A_{n-1}dt)\Psi_{n-1} = R_{n-1,n}\Psi_{n-1} \tag{18}$$

Hence the mean variance is given by

$$\mathbb{E}(\Psi_n) = \mathbb{E}(R_{0,n}\Psi_0) + 0 \tag{19}$$

$$\mathbb{E}(<\Psi_n - \mathbb{E}(\Psi_n), \Psi_n - \mathbb{E}(\Psi_n)>) = dt\sum_{k=0}^{n-2}\sum_{j=0}^{n-2}\mathbb{E}(\underbrace{< R_{k+1,n}\zeta_k, R_{j+1,n}\zeta_j >}_{<R^T_{j+1,n}, R_{k+1,n}\zeta_k,\zeta_j>}) \tag{20}$$

Now let

$$B = dt \sum_{k=0}^{n-2} \sum_{j=0}^{n-2} R_{j+1,n}^T R_{k+1,n} \delta j, k dt \,, \tag{21}$$

so that the total covariance is

$$N = tr(BC) = \sum_{i,j} \underbrace{\lambda_i \mu_j}_{\text{eigenvalues of B (resp C)}} | < \underbrace{\underline{v_i}, \underline{w_j}}_{\text{eigenvectors of B (resp C)}} > |^2 \tag{22}$$

where C is the covariance matrix. The first eigenvector of B is called the stochastic optimum. The use of this eigenvector as a forcing pattern triggers the maximum response from the model.