Lecture 10: Dansgaard-Oeschger Events (continued) & Ocean Western Boundary Current Variability

Henk A. Dijkstra; notes by Tom Beucler & Andre Souza

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1 Stochastic Resonance

It is possible to synchronize transitions when you have a periodic forcing on a double potential well:

$$V(x) = -\frac{x^2}{2} + \frac{x^4}{4} - \epsilon x \cos(\Omega\tau)$$

Note that the fixed points location (defined by $V'(x) = 0$ and $V''(x) > 0$) do not strongly vary with τ if the amplitude ϵ is small (in which case the fixed points are given by $x_\pm \approx \pm 1$), unlike the value of the potential V at this fixed points. For $\epsilon \ll 1$, these two values are approximately given by:

$$V(x_\pm) \approx V(\pm 1) = -[\frac{1}{4} \pm \epsilon \cos(\Omega\tau)]$$

Using the Laplace's approximation, the transition times are approximately given by:

$$\boxed{< t_{-1\rightarrow 1} > \approx 2\pi\sqrt{\frac{1}{-V''(0)V''(-1)}} \exp\{\frac{2[V(0) - V(-1)]}{\sigma^2}\} \approx \sqrt{2}\pi \exp[\frac{1 - 4\epsilon\cos(\Omega\tau)}{2\sigma^2}]}$$

$$\boxed{< t_{1\rightarrow -1} > \approx 2\pi\sqrt{\frac{1}{-V''(0)V''(1)}} \exp\{\frac{2[V(0) - V(1)]}{\sigma^2}\} \approx \sqrt{2}\pi \exp[\frac{1 + 4\epsilon\cos(\Omega\tau)}{2\sigma^2}]}$$

The transition times vary with τ as the potential changes shape. Because the variance in the transition time is very small compared to the transition time itself, the transition occurs over a small time-interval. As a consequence, the Fourier spectrum has a strong peak at the forcing frequency Ω. In the case of a small periodic forcing ϵ, the synchronization can occur for moderate values of σ.

For example, if we take the small amplitude to be $\epsilon = 0.1$, the shape of the potential is very close to a double well. If we suppose that at $\tau = 0$, the state of the system is near $x_+ \approx 1$, the transition time $< t_{1\rightarrow -1} >$ at this τ is maximal, as the potential well is deepest. If $\frac{\pi}{\Omega} \ll < t_{1\rightarrow -1} > (\tau = 0)$, then the well will change shape and the system will almost surely exit the well at $\tau = \frac{\pi}{\Omega}$ where the mean escape time $< t_{1\rightarrow -1} >$ is minimal:

The same reasoning can be applied when the system starts near $x_- \approx -1$ at $\tau = \frac{\pi}{\Omega}$. Thus, for small amplitude, the transitions of the system approximately occur when τ is a

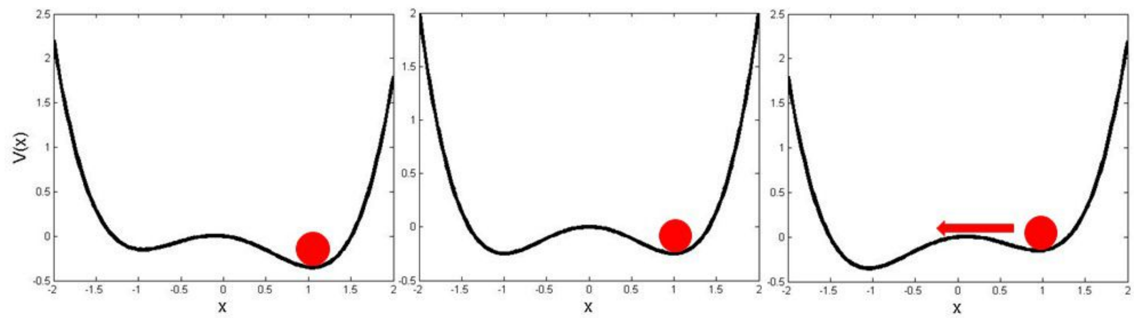

Figure 1: $V(x)$ vs x for $\tau = 0$ (left) $\tau = \frac{\pi}{2\Omega}$ (middle) and $\tau = \frac{\pi}{\Omega}$ (right). The state of the system is represented by the red dot.

multiple of $\frac{\pi}{\Omega}$, and the system stochastically resonates with the forcing of angular frequency Ω:

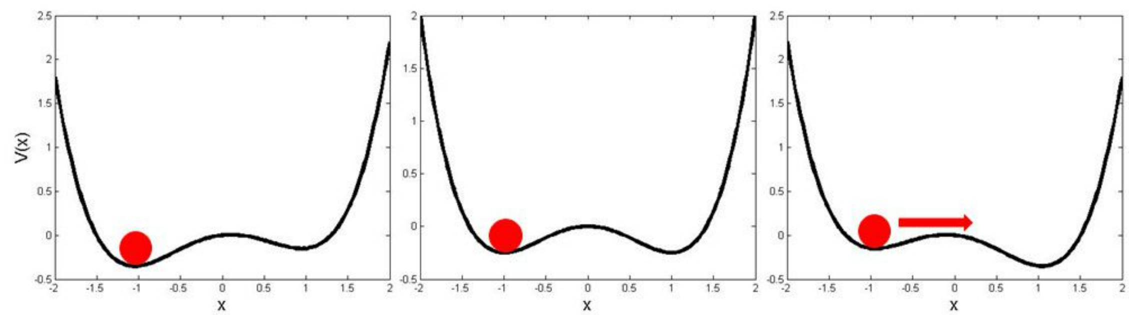

Figure 2: $V(x)$ vs x for $\tau = \frac{\pi}{\Omega}$ (left) $\tau = \frac{3\pi}{2\Omega}$ (middle) and $\tau = \frac{2\pi}{\Omega}$ (right). The state of the system is represented by the red dot.

The synchronization of transitions has applications in climatic models: If a periodic forcing is imposed, it is possible to have synchronization in the transitions, which allows for the existence of multiple equillibria. However, attributing this signal to stochastic resonance is a controversial topic. For instance, stochastic resonance has been suggested as a mechanism for the Dansgaard-Oeschger events, but many questions/doubts remain:

- Did multiple states of the Meridional Overturning Circulation exist during glacial times?
- What is the origin of the 1500 years period in the freshwater forcing?

2 Ocean Western Boundary Current Variability

2.1 Chaotic behavior of the Kuroshio current

We are now interested in systems that exhibit chaotic behaviors, and ask ourselves what to do when the circulation is so hard to model directly? Looking at the Sea Surface Height

(SSH) variability, we can see that it is predominant in the Western boundaries of the basins. In this lecture, we will focus on the Kuroshio current in Japan. Qiu et al. (2005) have observed a strong interannual-decadal time scale transitions between different Kuroshio paths (which look like spaghetti on the SSH contours), referred as path transitions. Looking at the variation of the path length in time, you have a strong variability: we start with a low path length, and then suddenly see a much bigger path lengths, corresponding to a meandric current. In this case, due to the lack of data, it is hard to test for red noise; instead they use a very simple reduced gravity shallow-water model for the wind:

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \underline{\nabla})\underline{u} + f\underline{k} \wedge \underline{u} = -g' \underline{\nabla \eta} + A_H \nabla^2 \underline{u} + \frac{\underline{\tau}}{\rho h} - \gamma |\underline{u}| \underline{u}$$

$$\frac{\partial h}{\partial t} + \underline{\nabla} \cdot (h\underline{u}) = 0$$

Taking a few points along the coast line, the wind profile $\underline{\tau}$ is mimicked. As it is very difficult to measure/determine the lateral friction A_H, it is taken as a control parameter. For $A_H = 220 m^2.s^{-1}$, Pierini et al. (2009) notice a strong transition in the behavior of the model, which does not correspond to the observations anymore, if metrics such as SSH are considered. The strong transition in the behavior of the system leads us to think that a nonlinear transition occurs when A_H becomes large enough.

2.2 Deterministic quasi-geostrophic barotropic model

To model this transition, we adopt a Quasi-geostrophic barotropic model, giving us an evolution equation for the streamfunction ψ:

$$\boxed{\frac{\partial \zeta}{\partial t} + \underline{u} \cdot \underline{\nabla \zeta} + \beta \frac{\partial \psi}{\partial x} = \frac{\nabla^2 \zeta}{Re} + \alpha \underline{k} \cdot (\underline{\nabla} \wedge \underline{\tau})}$$

where the horizontal velocity $\underline{u}$ and the relative vorticity ζ are related to the streamfunction ψ by:

$$\underline{u} = \underline{k} \wedge \underline{\nabla \psi}$$

$$\zeta = \nabla^2 \psi$$

We choose a double-gyre wind stress:

$$\underline{\tau} = -\cos(2\pi y)\underline{i}$$

so that we have eastwards wind in the Northern part of the domain and westwards wind in the Southern part of the domain. The control parameters is now the Reynolds number of the flow:

$$Re = \frac{UL}{A_H} \propto A_H^{-1}$$

Note that we have chosen the wind field to be symmetrical about the mean axis of the domain, which implies meridional symmetry of the equations of motion about this axis. We thus observe a Pitchfork bifurcation as the Reynolds number exceeds its critical value, which is the only co-dim 1 bifurcation leading to symmetry breaking. The flow can indeed break from a double-gyre configuration to:

- Steady-state streamfunction patterns.
- A jet-down configuration.
- A jet-up configuration.

At low-frequency, the two eigenmodes of the system (P and L) can merge to give interesting periods depending on the choice of the parameters in the model; for instance, the period of the gyre mode at Hopf bifurcation ($Re \sim 80$) is approximately 1.5y whereas the period of other Rossby-basin modes is much shorter, of order 2-4 months. Increasing Re and looking at the bifurcation diagram in (Re, $\frac{\psi_{min}+\psi_{max}}{|\psi_{maix}|}$) space, the deterministic system undergoes in this order:

- A Pitchfork bifurcation.
- A Hopf bifurcation.
- Gyre modes then appear.
- A homoclinic orbit then appears, connecting back the system to its initial fixed point.

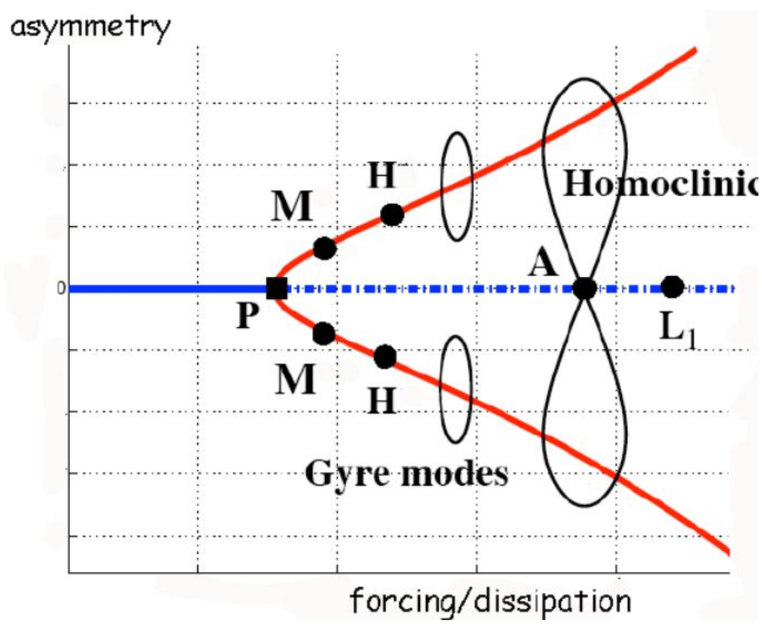

Figure 3: Schematic bifurcation diagram for the double-gyre model

The central question of this lecture is then: What happens if noise is added to the system? In the case of a chaotic system, such as the previous one, the PDEs are too hard to tackle directly, and we thus need to derive a low-order model. We:

- Neglect the diffusive term $\frac{1}{Re}\nabla^2\zeta$ and replace it with a Rayleigh drag $-\mu\zeta$.
- Choose the wind-stress strength as the control parameter.

We then truncate the system by focusing on the 4 first modes of the basin in the meridional direction:

$$\boxed{\psi(x,y,t) = \sum_{k=1}^{4} A_k(t) G(x) \sin(ky)}$$

where:

$$G_s(x) = \exp(-sx) \cdot \sin x$$

and $(x, y) \in [0, \pi]^2$. If we Galerkin project the equations for ψ, we obtain a low-order model, consisting of 4 ODEs for the amplitudes $A_k(t)$:

$$\frac{dA_1}{dt} = c_1(A_1A_2 + A_2A_3 + A_3A_4) - A_1$$

$$\frac{dA_2}{dt} = 2c_2(A_1A_3 + A_2A_4) - c_2A_1^2 - A_2 + c_5\alpha$$

$$\frac{dA_3}{dt} = c_3A_1(A_4 - A_2) - A_3$$

$$\frac{dA_4}{dt} = -c_4A_2^2 - 2c_4A_1A_3 - A_4$$

A wind-stress amplitude $\tau_0 = 0.1Pa$ gives $\alpha = 20$. It is possible to prove that this low-order model exhibits a transition sequence as α increases:

- First, we have a Pitchfork bifurcation.
- Then, we have a Hopf bifurcation.
- Finally, a homoclinic orbit appears.

Numerically, we show that the previous system has a steady attractive set with a first Lyapunov exponent $\lambda_1 > 0$.

2.3 Stochastic low-order models for chaotic systems

We are now able to add noise in the system, by transforming the previous system of ODEs in a system of SDEs:

$$dA_1 = [c_1(A_1A_2 + A_2A_3 + A_3A_4) - A_1]dt$$

$$dA_2 = [2c_2(A_1A_3 + A_2A_4) - c_2A_1^2 - A_2]dt + c_5\alpha(dt + \sigma \circ dW)$$

$$dA_3 = [c_3A_1(A_4 - A_2) - A_3]dt$$

$$dA_4 = [-c_4A_2^2 - 2c_4A_1A_3 - A_4]dt$$

Note that adding noise to a dynamical system changes the definition of what we call an attractor. We study the very simple 1D example:

$$\begin{cases} \dot{x} = -x + t \\ x(s) = x_0 \end{cases}$$

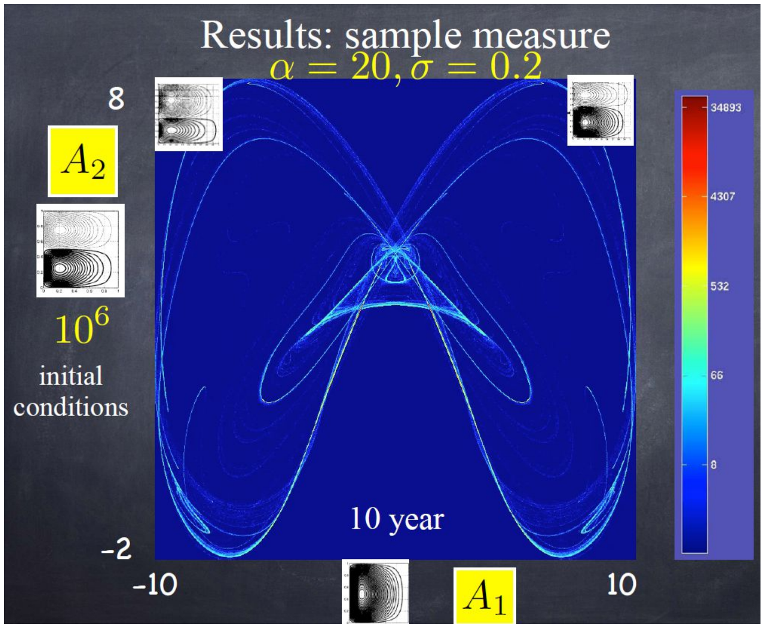

Figure 4: Numerical simulation of the low-order dynamical system with $N = 10^6$ initial conditions for a wind-stress $\tau = 0.1hPa$

Adding noise to this equation will not change its monotony. The solution of this system is given by:

$$x(t) = \exp(s - t) \cdot [x_0 - s + 1] + t - 1$$

If we wanna look at the attractor in terms of asymptotic limit, we start by noticing that all the trajectories eventually go to $t - 1$. If you consider the following flow:

$$\varphi(t, s)[x] = \exp(s - t) \cdot [x - s + 1] + t - 1$$

you are forced to go backwards in time and define an attractor differently.

$$\boxed{\lim_{s \to -\infty} |\varphi(t, s)[x] - A(t)| \to 0}$$

where:

$$A(t) = t - 1$$

Indeed, the noise might make the system change attractor, which prevents us of going forward in time to define the attractor. Similarly to what we did for the previously derived low-order model, we can add noise to the Lorenz model:

$$dX = s(Y - X)dt + \sigma X dW$$

$$dY = (rX - Y - XZ)dt + \sigma Y dW$$

$$dZ = (-bZ + XY)dt + \sigma Z dW$$

Numerically, we initiate the system with many initial conditions (typically $N \sim 10^6 - 10^9$) and perturb each initial condition with a single noise realization. We then look at the density of trajectories in phase space. To study the effects of noise in the wind-stress forcing on the intrinsic variability in this PDE model, three methods exist:

1. Study the local PDF through linearized dynamics (cf Kuehn et al. 2012)
2. Use dynamical orthogonal field theory (cf Sapsis et al. 2009 and Sapsis et al. 2013)
3. Use non-Markovian model reduction techniques (cf Chekroun et al. 2015)

We will study the second method and apply it to our stochastic barotropic QG model:

$$\frac{\partial \underline{u}}{\partial t} = -\underline{\nabla p} + \frac{\nabla^2 \underline{u}}{Re} - (\underline{u} \cdot \underline{\nabla})\underline{u} - f\underline{k} \wedge \underline{u} + \underline{\tau_W}(\underline{x}, t)$$

$$\underline{\nabla} \cdot \underline{u} - 0$$

where we separate the wind in its deterministic double-gyre part and a stochastic part:

$$\underline{\tau_W} = \tau_{DG}(y)\underline{i} + \sigma\tau_{stochastic}(x, y, t)\underline{i}$$

2.4 The dynamical orthogonal field method

They dynamical orthogonal field equation can be derived following Sapsis and Lermusiaux (2009). Starting from a general SDE of the form:

$$\boxed{\frac{\partial \underline{u}(\underline{x}, t, \omega)}{\partial t} = \underline{\mathcal{L}}[\underline{u}(\underline{x}, t, \omega), \omega]}$$

where:

$$(\underline{x}, t, \omega) \in \mathcal{D} \times \mathcal{T} \times \Omega$$

We define the mean of the field $\underline{u}$ as:

$$\overline{u}(\underline{x}, t) = \mathbb{E}^{\omega}[\underline{u}(\underline{x}, t, \omega)] = \int_{(\Omega)} \underline{u}(\underline{x}, t, \omega) d\mathbb{P}(\omega)$$

where $\mathbb{P}$ is a probability measure on ω. From the covariance matrix:

$$\underline{\underline{C_{\underline{u}(\cdot,t)\underline{u}(\cdot,t)}}}(\underline{x}, \underline{y}) = \mathbb{E}^{\omega}[(\underline{u}(\underline{x}, t, \omega) - \underline{\overline{u}}(x, t))(\underline{u}(\underline{y}, t, \omega) - \underline{\overline{u}}(\underline{y}, t))^T]$$

we can define the integral operator:

$$\underline{\mathcal{T}_{\underline{\underline{C}}}}(\varphi) = \int_{(\mathcal{D})} \underline{\underline{C_{\underline{u}(\cdot,t)\underline{u}(\cdot,t)}}}(\underline{x}, \underline{y}) \cdot \underline{\varphi}(\underline{x}, t) d\underline{x}$$

and prove that it is compact, self-adjoint and positive. It follows that any random field $\underline{u}$ has a Karhuven-Loeve expansion at a given time t:

$$\underline{u}(\underline{x}, t, \omega) = \underline{\overline{u}}(\underline{x}, t) + \sum_{i=1}^{+\infty} Y_i(t, \omega)\underline{u_i}(\underline{x}, t)$$

where:

- $\underline{u_i}(\underline{x}, t)$ are the eigenfunctions of $\mathcal{T}_{\underline{\underline{C}}}$
- $Y_i(t, \omega)$ are zero-mean stochastic processes with variance $\mathbb{E}^\omega[Y_i^2(t, \omega)]$ which are eigenvalues of the following eigenvalue problem:

$$\mathcal{T}_{\underline{\underline{C}}}[\underline{u_i}(\underline{x}, t)] = \mathbb{E}^\omega[Y_i^2(t, \omega)]\underline{u_i}(\underline{y}, t)$$

In many physical problems of interest (including ours), $\mathbb{E}^\omega[Y_i^2(t, \omega)] \sim \exp(-ci)$ where $c > 0$, so that to a good approximation, we can truncate the Karhuven-Loeve expansion to a finite number of terms:

$$\boxed{\overline{u}(\underline{x}, t, \omega) = \overline{u}(\underline{x}, t) + \sum_{i=1}^{s} Y_i(t, \omega)\underline{u_i}(\underline{x}, t)}$$

We can see that the variation of the stochastic coefficients Y_i can express exclusively the evolution of the uncertainty within the stochastic space $V_S = span(\underline{u_i} | i \in [1, s])$. However, the evolution of the stochastic basis $\underline{u_i}$ itself allows the uncertainty to cover V_S and $V_S^\perp$. To avoid redundancy in the evolution of the uncertainty, we impose that the evolution of the basis $\underline{u_i}$ stays in $V_S^\perp$, ie:

$$\frac{dV_S}{dt} \perp V_S \;\; \Leftrightarrow < \frac{\partial}{\partial t}\underline{u_i} | \underline{u_j} >= 0$$

This is the dynamically orthogonal condition (DO condition). We can now derive the DO field equations by inserting the DO representation in the initial evolution equation:

$$\frac{\partial}{\partial t}\overline{u} + \frac{dY_i}{dt}\underline{u_i} + Y_i \frac{\partial}{\partial t}\underline{u_i} = \underline{\mathcal{L}}[\underline{u}(\underline{x}, t, \omega), \omega]$$

Applying $\mathbb{E}^\omega$ to the previous equation, we obtain an evolution equation for the mean part of the representation:

$$\boxed{\frac{\partial}{\partial t}\overline{u} = \mathbb{E}^\omega\{\mathcal{L}[\underline{u}(\underline{x}, t, \omega), \omega]\}}$$

Taking the inner product of the evolution equation with $\underline{u_j}$, applying the orthonormality condition of the $\underline{u_i}$, and the DO condition, we obtain:

$$\frac{dY_j}{dt} =< \frac{\partial}{\partial t}\overline{u} | \underline{u_j} >=< \underline{\mathcal{L}}(\underline{u}) | \underline{u_j} >$$

Applying $\mathbb{E}^\omega$ to the previous equation and using the evolution equation for the mean part of the representation, it is possible do derive an equation for the zero-mean stochastic processes Y_i:

$$\boxed{\frac{dY_i}{dt} =< \underline{\mathcal{L}}[\underline{u}, \omega] - \mathbb{E}^\omega\{\underline{\mathcal{L}}[\underline{u}, \omega]\} | \underline{u_i} >}$$

Finally, it is also possible to derive an equation for the basis vectors:

$$\boxed{\frac{\partial}{\partial t}\underline{u_i} = \underline{\Pi_{V_S^\perp}}[\mathbb{E}^\omega\{\underline{\mathcal{L}}[\underline{u}, \omega]Y_j\}] \cdot \underline{\underline{C^{-1}_{Y_i(t)Y_j(t)}}}}$$

where the operator $\underline{\Pi_{V_S^\perp}}$ is defined as:

$$\underline{\Pi_{V_S^\perp}}[\underline{F}(\underline{x})] = \underline{F}(\underline{x}) - < \underline{F}(\underline{x}) | \underline{u_k}(\underline{x}, t) > \underline{u_k}(\underline{x}, t)$$

2.5 Application to the quasi-geostrophic barotropic model

We come back to our BT QG model and apply the DO analysis following Sapsis and Dijkstra (2013). We first expand the horizontal velocity and the pressure:

$$\underline{u}(\underline{x}, t, \omega) = \overline{\underline{u}}(\underline{x}, t) + \sum_{i=1}^{s} Y_i(t, \omega)\underline{u_i}(\underline{x}, t)$$

$$p = p_0 + Y_i p_i - Y_i Y_j p_{ij} + Z_r b_r$$

where Z_r are the coefficients of the noisy part of the wind:

$$\sigma\tau_a(x, y, t) = \sum_{k=1}^{s} Z_k(t, \omega)\sigma_k(\underline{x}, t)$$

The DO mean equations can be written:

$$\boxed{\frac{\partial\overline{\underline{u}}}{\partial t} = -\underline{\nabla p_0} + \frac{\nabla^2\overline{\underline{u}}}{Re} - (\overline{\underline{u}} \cdot \underline{\nabla})\overline{\underline{u}} - f\underline{k} \wedge \overline{\underline{u}} + \underline{\tau_d}(\underline{x}, t) - \underline{\underline{C_{Y_iY_j}}} \cdot [-\underline{\nabla p_{ij}} + \frac{1}{2}(\underline{u_i} \cdot \underline{\nabla})\underline{u_j} + \frac{1}{2}(\underline{u_j} \cdot \underline{\nabla})\underline{u_i}]}$$

$$0 = \underline{\nabla} \cdot \overline{\underline{u}}$$

Projecting on the DO modes, it is possible to obtain a solvable system of (s+1) PDEs. The stochastic wind stress forcing follows the bulk formula for the momentum flux:

$$\underline{\tau_{stochastic}} = \rho_{air} C_D |\underline{u}'|\underline{u}'$$

where ρ_{air} is the air density, $\underline{u}'$ the near-surface wind's velocity and C_D the drag coefficient. The near-surface wind is taken to be stochastic:

$$\underline{u}' = f(x, y)\underline{\eta}(t)$$

where $\underline{\eta}(t)$ is a white/colored noise vector depending on the experiment, with mean 0 and variance σ. The weight function f parametrizes the spatial structure of the atmospheric variability with a Gaussian shape, whose origin is placed at the center of the basin:

$$f(x, y) = \alpha[\pi\lambda_x\lambda_y \text{erf}(\frac{L_x}{2\lambda_x})\text{erf}(\frac{L_y}{2\lambda_y})]^{-\frac{1}{2}} \exp(\frac{x^2}{2\lambda_x^2} + \frac{y^2}{2\lambda_y^2})$$

Looking at the resulting 3D contours of the PDF, we can see that the main effect of the noise is to allow for multiple equillibria of the system. Looking at the effect of the noise in more detail, we can see significant differences if the noise is chosen to be white or colored:

- If we choose a white-noise excitation $\underline{\eta}(t) \propto \underline{dW}(t)$ where $\underline{W}$ is a vectorial Wiener process, then the stochastic excitation has zero effect on the instantaneous evolution of the mean field, and the shape of the stochastic subspace V_S. As a consequence, it will not influence the general statistics of the double-gyre flow.

- More generally, we can choose a colored noise excitation, for instance through the Ornstein-Uhlenbeck process:

$$\tau d\underline{\eta}(t) = -\underline{\eta}(t) + \sqrt{2\tau}\underline{dW}(t)$$

where τ is the decorrelation time scale ($\tau \gg 1$ corresponds to the deterministic case and $\tau \ll 1$ to the white-noise case). This does not only allow to internally transfer energy between the DO modes, but it also allows the stochastic modes to directly absorb energy from the stochastic forcing. Depending on the value of τ, colored noise can either destabilize some DO modes and push the system in a statistically steady regime (small memory $\Leftrightarrow$ small τ), or reduce the complexity of the system by bringing it to a single unstable mode (long memory $\Leftrightarrow$ large τ).

3 Conclusion

Is summary, it is possible to add noise to a chaotic system and still solve for a low order equivalent system. We can see that the noise adds a lot of variability, helping us explore new flavors of the climatic system. To study the effect of the noise on a PDE system with more precision, we have introduced the method of the dynamical-orthogonal field. Under certain condition, this method reduces the analysis of a general continuous stochastic field to a finite number of orthonormal mode, which define a stochastic subspace where the solution lives. In the special case of the BT QG model , we have seen that white-noise has a minimal effect on the reduced dynamics, whereas colored noise can significantly change the behavior of the system.