Ray Theory of Nonlinear Water Waves

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September 29, 2009

1 Introduction.

In past decades there has been growing interest in propagation of nonlinear waves over a variable bottom and internal waves in a variable medium, which leads to a nonlinear partial differential equation of KdV type with variable coefficients (vc-KdV). The study of vc-KdV equation could enhance our understanding of many problems in oceanography and plasma, such as tsunami([10]) and dust ion acoustic waves([7]). A methodology in linear theory, the so-called ray method, was extended to nonlinear wave theory in Cartesian geometry by Shen&Keller[8] and in cylindrical geometry by Shen&Shen[9]. The wave amplitude satisfies the vc-KdV equation along each ray which is determined by the Eikonal equation. The solution is valid when the wavelength is small compared to other horizontal scale lengths. Since the canonical Korteweg-de Vries equation describes the propagation of long waves in shallow water, in some sense this method combines features of both the short wave theory and long wave theory.

In section 2 a vc-KdV equation is derived by applying the ray method in Cartesian coordinate system. Section 3 is dedicated to seek solutions to Eikonal equation in the case that the topography is independent of one spatial variable. Rays in plane waves and circular waves propagating over different bottoms are calculated and plotted. A variable coefficient KdV equation in cylindrical geometry (vc-cy-KdV) is given in section 4, which can be transformed into a vc-KdV by a suitable change of variables. Exact solutions to the vc-cy-KdV equation and conditions under which we are able to find them explicitly are discussed. We also obtain approximate solutions using the method developed by Grimshaw[4], and study the large-time behavior of the amplitude of both cnoidal waves and solitary waves. These two types of waves share the same critical value of the bottom slope for the amplitude to blow up or vanish far away from the source of waves. Finally in section 5 we discuss some directions in which future work could advance.

2 Formulation of the problem.

Let us consider an inviscid, incompressible fluid over an uneven bottom $z^* = -h^*(x^*, y^*)$, as sketched in figure 1. The free surface is denoted by $z^* = \eta^*(x^*, y^*, z^*, t^*)$ where η^* is an unknown function. Denote by ρ, P and $\vec{u}$ the fluid density, pressure and velocity respectively. We can define dimensionless variables as follows:

$$
\begin{aligned}
h &= \frac{h^*}{H}, \quad \eta = \frac{\eta^*}{H}, \quad \epsilon = (\frac{H}{L})^{\frac{2}{3}}, \quad \rho = \frac{\rho^*}{D} \\
(x, y, z) &= (\frac{\epsilon^{\frac{3}{2}}}{H}x^*, \frac{\epsilon^{\frac{3}{2}}}{H}y^*, \frac{z^*}{H}), \quad t = \epsilon^{\frac{3}{2}}(\frac{g}{H})^{\frac{1}{2}}t^*, \\
P &= \frac{P^*}{gHD}, \quad \vec{u} = (u, v, w) = (gH)^{-\frac{1}{2}}(u^*, v^*, \epsilon^{-\frac{1}{2}}w^*),
\end{aligned}
\tag{2.1}
$$

where H, L, D are scales for vertical length, horizontal length and density respectively.

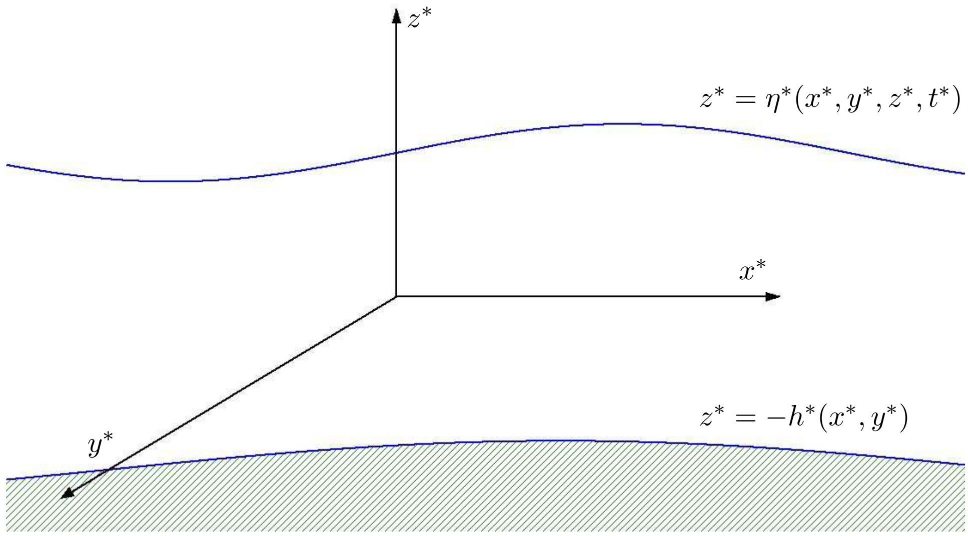


Figure 1: Flow region is between the rigid bottom $z^* = -h^*(x^*, y^*)$ and the free surface $z^* = \eta^*(x^*, y^*, z^*, t^*)$.

Using these dimensionless variables, we can represent the equation of motion, equation of continuity and boundary conditions on the rigid bottom and the free surface in the following way:

- Equation of motion: $\rho^*(\frac{\partial \vec{u}^*}{\partial t^*} + \vec{u}^* \cdot \nabla^* \vec{u}^*) = -\nabla^* P^*$ becomes

$$\epsilon[\rho(u_t + uu_x + vu_y) + P_x] + \rho w u_z = 0, \tag{2.2}$$

$$\epsilon[\rho(v_t + uv_x + vv_y) + P_y] + \rho w v_z = 0, \tag{2.3}$$

$$\epsilon^2 \rho(w_t + uw_x + vw_y) + \epsilon \rho w w_z + P_z + \rho = 0. \tag{2.4}$$

- Equation of continuity $\nabla^* \cdot \vec{u}^* = 0$ becomes

$$\epsilon(u_x + v_y) + w_z = 0. \tag{2.5}$$

- Two boundary conditions $P^* = const$, $\frac{D^*}{D^* t^*}(z^* - \eta^*) = 0$ at $z^* = \eta^*$ and $\vec{u}^* \cdot \nabla^*(z^* +$

$h^*) = 0$ at $z^* = -h^*$ becomes

$$P = C, \quad \text{at } z = \eta, \tag{2.6a}$$

$$\epsilon(\eta_t + u\eta_x + v\eta_y) - w = 0, \quad \text{at } z = \eta, \tag{2.6b}$$

$$\epsilon(uh_x + vh_y) + w = 0, \quad \text{at } z = -h. \tag{2.6c}$$

Here ∇^* is the dimensional gradient operator and $\frac{D^*}{D^* t^*}$ is the dimensional material derivative $\frac{D}{Dt}$.

2.1 Asymptotic expansion and Eikonal equation.

To find wave-like solutions to (2.2)-(2.6) we introduce a *phase function* $S(x, y, t)$ and a *"fast-phase"* variable $\xi = \epsilon^{-1} S(x, y, t)$. Assume that u, v, w, P and η all have the form of asymptotic expansion

$$G(t, x, y, z, \xi) = G_0(t, x, y, z, \xi) + \epsilon G_1(t, x, y, z, \xi) + O(\epsilon^2). \tag{2.7}$$

By noting that operators $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ become $\frac{\partial}{\partial t} + \epsilon^{-1} S_t \frac{\partial}{\partial \xi}$, $\frac{\partial}{\partial x} + \epsilon^{-1} S_x \frac{\partial}{\partial \xi}$ and $\frac{\partial}{\partial y} + \epsilon^{-1} S_y \frac{\partial}{\partial \xi}$ respectively, we can expand equations (2.2)-(2.6). Equating $O(1)$ terms to zero yields the static-flow solution; and equating $O(\epsilon)$ terms to zero yields

$$\rho u_{1\xi} S_t + P_{1\xi} S_x = 0, \tag{2.8}$$

$$\rho v_{1\xi} S_t + P_{1\xi} S_y = 0, \tag{2.9}$$

$$P_{1z} = 0, \tag{2.10}$$

$$u_{1\xi} S_x + v_{1\xi} S_y + w_{1z} = 0, \tag{2.11}$$

$$P_1 = \rho \eta_1, \quad \text{at } z = 0, \tag{2.12}$$

$$w_1 = \eta_{1\xi} S_t, \quad \text{at } z = 0, \tag{2.13}$$

$$w_1 = 0, \quad \text{at } z = -h(x, y). \tag{2.14}$$

To solve equations (2.8)-(2.14) for unknowns $u_1, v_1, w_1, P_1, \eta_1$ and S, we firstly solve (2.8) and (2.9) for $u_{1\xi}$ and $v_{1\xi}$ to get

$$u_{1\xi} = -\frac{P_{1\xi} S_x}{\rho S_t}, \quad v_{1\xi} = -\frac{P_{1\xi} S_y}{\rho S_t}. \tag{2.15}$$

Substituting (2.15) into (2.11) yields

$$w_{1z} = \frac{P_{1\xi}(S_x^2 + S_y^2)}{\rho S_t}. \tag{2.16}$$

From (2.10) and (2.16) we know w_{1z} is independent of z, so using boundary condition (2.14) we have

$$w_1 = \frac{P_{1\xi}(S_x^2 + S_y^2)}{\rho S_t}(z + h). \tag{2.17}$$

Equations (2.12) and (2.13) implies

$$w_1 = \frac{P_{1\xi} S_t}{\rho}, \quad \text{at } z = 0. \tag{2.18}$$

Now combining (2.17) and (2.18) and eliminating w_1 we get the *Eikonal* equation

$$\frac{S_t^2}{S_x^2 + S_y^2} = h(x, y). \tag{2.19}$$

By solving the Eikonal equation with proper boundary conditions we can determine the phase function $S(x, y, t)$, which is postponed to section 3. Once the surface elevation η_1 and the phase function S are determined, P_1 can be found by (2.12), and w_1 by (2.17). From (2.15) we can obtain special solutions u_1, v_1 as

$$u_1 = -\frac{P_1 S_x}{\rho S_t}, \quad v_1 = -\frac{P_1 S_y}{\rho S_t}. \tag{2.20}$$

Arbitrary functions independent of ξ can be added to the right-hand sides of (2.20). We choose them to be zeros so that the fluid velocity has the same direction as rays.

2.2 Amplitude equation.

To determine η_1, we need to proceed further by equating coefficients of $O(\epsilon^2)$ terms in equations (2.2)-(2.6):

$$S_x P_{2\xi} + \rho S_t u_{2\xi} = -(\rho u_{1t} + \rho u_1 S_x u_{1\xi} + \rho v_1 S_y u_{1\xi} + P_{1x}), \tag{2.21}$$

$$S_y P_{2\xi} + \rho S_t v_{2\xi} = -(\rho v_{1t} + \rho u_1 S_x v_{1\xi} + \rho v_1 S_y v_{1\xi} + P_{1y}), \tag{2.22}$$

$$P_{2z} = -\rho S_t w_{1\xi}, \tag{2.23}$$

$$w_{2z} + S_x u_{2\xi} + S_y v_{2\xi} = -u_{1x} - v_{1y}, \tag{2.24}$$

$$P_2 - \rho \eta_2 = 0, \quad \text{at } z = 0, \tag{2.25a}$$

$$w_2 - S_t \eta_{2\xi} = \eta_{1t} + u_1 S_x \eta_{1\xi} + v_1 S_y \eta_{1\xi} - w_{1z} \eta_1, \quad \text{at } z = 0, \tag{2.25b}$$

$$w_2 = -u_1 h_x - v_1 h_y, \quad \text{at } z = -h(x, y). \tag{2.25c}$$

Equations (2.21)-(2.24) are a system of equations of u_2, v_2, w_2, η_2 and P_2 whose right-hand sides are all known except for exactly one quantity (η_1, w_1, or P_1). By straightforward but a little tedious calculations we can eliminate u_2, v_2 and η_2 to get a compatibility condition under which a solution exists(see appendix in [8] for details). This compatibility condition turns out to be in a following KdV-like form of a nonlinear partial differential equation

$$\widehat{A}_t + \frac{J_t}{2J}\widehat{A} + (\frac{\omega h^2}{2} - \frac{\omega}{\rho^2 h})\widehat{A}\widehat{A}_\xi + \frac{\omega^3 h}{6}\widehat{A}_{\xi\xi\xi} = 0, \tag{2.26}$$

where $\widehat{A} = \rho \eta_1$, ω (angular frequency) will be defined in section 3, and J is the Jacobian from the ray coordinates $(\tau, \gamma_1, \gamma_2)$ to the time-Cartesian coordinate system (t, x, y).

Denote $A = J^{1/2}\widehat{A}$, then (2.26) becomes

$$A_t + f_1(t)AA_\xi + f_2(t)A_{\xi\xi\xi} = 0, \tag{2.27}$$

where $f_1(t) = J^{-1/2}(\frac{\omega h^2}{2} - \frac{\omega}{\rho^2 h})$, and $f_2(t) = \frac{\omega^3 h}{6}$.

Equation (2.27) is called the variable coefficient Korteweg-de Vries equation(vc-Kdv) and was derived by Shen&Keller[8]. It was shown by Joshi [5] that it is completely integrable if and only if coefficient functions f_1 and f_2 satisfy

$$f_2(t) = f_1(t)(c_1 \int^t f_1(s)ds + c_2), \tag{2.28}$$

for arbitrary constants c_1 and c_2. Under this condition, Lax pair and infinitely many conservation laws of (2.27) can be found(see e.g. Fan[2] and Zhang[12]).

3 Ray tracing.

Eikonal equation (2.19) can be solved by using the method of characteristics. By doing so, we obtain the following system of ordinary differential equations

$$\dot{x}(\tau) = -\frac{S_x}{\sqrt{S_x^2 + S_y^2}}\ \theta, \tag{3.1a}$$

$$\dot{y}(\tau) = -\frac{S_y}{\sqrt{S_x^2 + S_y^2}}\ \theta, \tag{3.1b}$$

$$\dot{t}(\tau) = 1, \tag{3.1c}$$

and

$$\dot{S}_x(\tau) = \sqrt{S_x^2 + S_y^2}\ \theta_x, \tag{3.2a}$$

$$\dot{S}_y(\tau) = \sqrt{S_x^2 + S_y^2}\ \theta_y, \tag{3.2b}$$

$$\dot{S}_t(\tau) = 0, \tag{3.2c}$$

and very importantly,

$$\dot{S}(\tau) = 0, \tag{3.3}$$

where $\theta(x, y) = -h^{\frac{1}{2}}(x, y)$, and *dot* is the total derivative with respect to τ.

System of equations (3.1) determines a two-parameter family of space-time curves, which are called *rays*; (3.2) determine the change of gradient of S along a ray; and (3.3) indicates that S remains constant along a ray. In following discussion we often do not distinguish ∂t and $\partial\tau$ due to (3.1c) unless special attention is drawn. It follows from (3.2c) that on each ray $S_t(\tau)$ is also a constant. We will only consider in this paper the case when S_t is the same constant for all rays. We call $\omega = -S_t$ the *angular frequency*.

We can parameterize rays by using parameters γ_1 and γ_2 which are constants along a ray. They can be chosen to be, for instance,

$$\begin{aligned} \gamma_1 &= S(x, y, t), \\ \gamma_2 &= y - y(\tau(x)), \end{aligned} \tag{3.4}$$

then $\frac{\partial \gamma_1}{\partial x} = S_x$, $\frac{\partial \gamma_1}{\partial y} = S_y$, $\frac{\partial \gamma_2}{\partial x} = -\frac{\dot{y}}{\dot{x}}$, and $\frac{\partial \gamma_2}{\partial y} = 1$. The Jacobian J from the ray coordinates $(\tau, \gamma_1, \gamma_2)$ to the time-Cartesian coordinate (t, x, y) can be calculated as

$$J = \left| \frac{\partial \gamma_1}{\partial x} \frac{\partial \gamma_2}{\partial y} - \frac{\partial \gamma_1}{\partial y} \frac{\partial \gamma_2}{\partial x} \right|^{-1} = \frac{S_x}{S_x^2 + S_y^2} = \frac{S_x h}{S_t^2}. \tag{3.5}$$

In general the system of ordinary differential equations (3.1)-(3.3) is difficult to solve since they are coupled with each other. But if the topography is only dependent on one variable they can be solved analytically. Assume $h_y = 0$, then it follows from (3.2b) that $S_y(\tau) = S_y(0) = k_2$. We substitute $\theta_x = \frac{\dot{\theta}}{\dot{x}}$ into (3.2a) and use (3.1a) to get

$$\frac{S_x \dot{S}_x}{S_x^2 + k_2^2} = -\frac{\dot{\theta}}{\theta}$$

Integrating it from 0 to τ gives us

$$S_x(\tau) = [h_0(k_1^2 + k_2^2)h^{-1} - k_2^2]^{\frac{1}{2}}, \tag{3.6}$$

where $k_1 = S_x(0)$ and $h_0 = h(0)$. The location of rays can be obtained by substituting $S_x(\tau)$ into (3.1a) and (3.1b):

$$\dot{x}(\tau) = \sqrt{h(x(\tau)) - \frac{k_2^2}{h_0(k_1^2 + k_2^2)} h^2(x(\tau))}, \tag{3.7a}$$

$$\dot{y}(\tau) = \frac{k_2}{\sqrt{h_0(k_1^2 + k_2^2)}} h(x(\tau)). \tag{3.7b}$$

Equation (3.7a) is a first-order ordinary differential equation thus can be solved analytically for a large scope of bottom functions $h(x)$; then (3.7b) is solved readily. The value of S at any point P will be given by its value at a boundary point Q from which the ray emits and passes through P.

Next we will consider several physical scenarios (various topography & various forms of waves) when (3.7) is solved with suitable boundary conditions and rays are calculated explicitly.

3.1 Plane waves.

Let us suppose that a sinusoidal plane wave is generated at $x = -\infty$ and propagating towards the shore at $x = l$. The angle between the wavefronts and positive $y-$axis is φ.

The bottom function is selected in the following way so that (3.2a) and (3.2b) can be solved easily,

$$h(x) = \begin{cases} \alpha^2 l^2, & x \leq 0; \\ \alpha^2 (l-x)^2, & x \geq 0. \end{cases} \quad \text{with } \alpha > 0. \tag{3.8}$$

A phase function representing sinusoidal plane waves is $S(x, y, t) = k_1 x + k_2 y - \omega t$ with $\frac{k_2}{k_1} = \arctan(\varphi)$. Parameters k_1, k_2 and ω are related by the dispersion relation $\frac{\omega^2}{k_1^2+k_2^2} = h_{loc}$, where h_{loc} is the local depth of water. The boundary conditions for the system are prescribed on $\Gamma_1 = \{(x, y, z)|x = 0, y \in \mathbb{R}, t > 0\}$ as

$$S_x(0) = k_1, \quad S_y(0) = k_2, \quad S_t(0) = -\omega, \quad S(0, y, t) = k_2 y - \omega t. \tag{3.9}$$

It immediately follows from (3.2) that $S_y(\tau) = k_2$, $S_t(\tau) = -\omega$, and

$$S_x(\tau) = \frac{1}{2c_1}(c_1^2 e^{\alpha\tau} - k_2^2 e^{-\alpha\tau}), \tag{3.10}$$

where the constant c_1 is defined as $c_1 = \sqrt{k_1^2 + k_2^2} + k_2$. The location of characteristics can be readily calculated

$$\begin{aligned} \dot{x} &= \alpha(l-x)\frac{c_1^2 e^{\alpha\tau} - k_2^2 e^{-\alpha\tau}}{c_1^2 e^{\alpha\tau} + k_2^2 e^{-\alpha\tau}} \\ \dot{y} &= \frac{\alpha l(c_1^2 + k_2^2)}{2c_1 k_2}[\text{sech}(\log\frac{c_1}{k_2} + \alpha\tau)]^2 \\ \dot{t} &= 1. \end{aligned} \tag{3.11}$$

Using initial conditions $x(0) = 0$, $y(0) = y_0$ and $t(0) = t_0$ we can integrate (3.11) to obtain

$$\begin{aligned} x(\tau) &= l(1 - \frac{c_1^2 + k_2^2}{c_1^2 e^{\alpha\tau} + k_2^2 e^{-\alpha\tau}}) \\ y(\tau) &= y_0 + \frac{l(c_1^2 + k_2^2)}{2c_1 k_2}(\frac{c_1^2 e^{\alpha\tau} - k_2^2 e^{-\alpha\tau}}{c_1^2 e^{\alpha\tau} + k_2^2 e^{-\alpha\tau}} - \frac{c_1^2 - k_2^2}{c_1^2 + k_2^2}), \\ t(\tau) &= t_0 + \tau, \end{aligned} \tag{3.12}$$

which are plotted in figure 2(a).

To find the phase function S at any point (x^*, y^*, t^*), we need to find y_0, t_0 and τ such that $x(\tau) = x^*$, $y(\tau) = y^*$, and $t(\tau) = t^*$. Then the phase function is

$$S(x^*, y^*, t^*) = S(0, y_0, t_0) = k_2 y_0 - \omega t_0.$$

One can easily see in (3.12) that

$$\lim_{\tau\to\infty} x(\tau) = l, \quad \text{and} \quad \lim_{\tau\to\infty} y(\tau) = y_0 + (\sqrt{k_1^2 + k_2^2} + k_1)^{-1} k_2 l,$$

indicating that the rays will approach the shoreline asymptotically.

3.2 Waves generated by a point oscillator.

Suppose that there is a point source oscillator at the origin, generating sinusoidal waves. In this case all information on the phase function are only available on a straight line $\Gamma_2 = \{x = y = 0, t \geq 0\}$, where all rays are emitted. Therefore in order to determine value of the phase function it is necessary to find both the point on $t-$axis and the angle from which a ray emits. Since in a small neighborhood of the origin the phase function can be written in the form of $S = kr - \omega t + o(r)$, the boundary condition for phase function on $t-$axis is $S = -\omega t$. For a ray leaving the origin with an angle ϕ to the positive $x-$axis, its initial condition is $(S_x, S_y, S_t)|_{x=y=0} = (k\cos\phi, k\sin\phi, -w)$.

The analysis leading to the result in previous section can be applied here, with k_1, k_2 replaced by $k\cos\phi$, $k\sin\phi$ respectively. A straightforward calculation shows that the rays are given by

$$\begin{aligned}
x(\tau) &= l - l(\cos^2(\frac{\phi}{2})e^{\alpha\tau} + \sin^2(\frac{\phi}{2})e^{-\alpha\tau})^{-1}, \\
y(\tau) &= \frac{l}{\sin\phi}(\frac{\cos^2(\frac{\phi}{2})e^{\alpha\tau} - \sin^2(\frac{\phi}{2})e^{-\alpha\tau}}{\cos^2(\frac{\phi}{2})e^{\alpha\tau} + \sin^2(\frac{\phi}{2})e^{-\alpha\tau}} - \cos\phi), \\
t(\tau) &= t_0 + \tau,
\end{aligned} \tag{3.13}$$

for $\phi \in (\frac{-\pi}{2}, \frac{\pi}{2})\backslash\{0\}$. Notice that $\lim_{\tau\to\infty} x(\tau) = l$, showing that rays approach the shore asymptotically which is similar to the plane-wave case. When $\phi \in (\frac{\pi}{2}, \frac{3\pi}{2})$, waves are traveling in a medium with constant water depth, thus all rays in this region are propagating radially. Rays given by (3.13) are plotted in figure2(b).

To find the value of S at any point (x^*, y^*, t^*), we need to find ϕ, t_0 and τ such that $x(\tau) = x^*$, $y(\tau) = y^*$ and $t(\tau) = t^*$. Then the phase function is

$$S(x^*, y^*, t^*) = S(0, 0, t_0) = -\omega t_0.$$

3.3 Reflected rays in plane waves.

The fact that rays approach to shoreline asymptotically without being reflected is due to the nature of varying topography. Indeed it can be easily shown that the rays will not be reflected back for a bottom function $h(x) \sim (x - l)^{-\beta}$ if and only if $\beta \geq 2$.

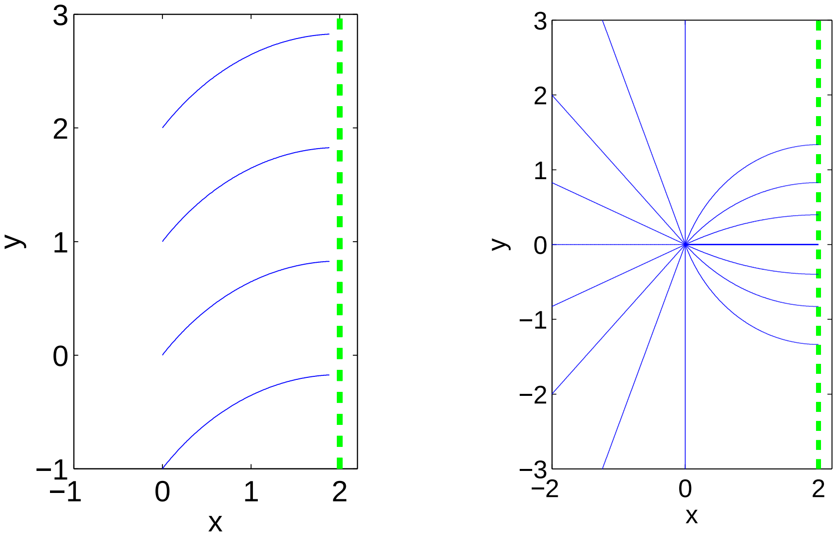
In the case of a piecewise linear topography:

$$h(x) = \begin{cases} \alpha^2 l, & x \leq 0; \\ \alpha^2(l - x), & x \geq 0. \end{cases} \quad \text{with } \alpha > 0, \tag{3.14}$$

we can solve equations (3.7a) and (3.7b) to get

$$x(\tau) = l - \frac{h_0(k_1^2 + k_2^2)}{2k_2^2}[1 - \sin(\frac{k_2\alpha}{\sqrt{h_0(k_1^2 + k_2^2)}}\tau + \arcsin(\frac{k_1^2 - k_2^2}{k_1^2 + k_2^2}))],$$

$$y(\tau) = y_0 - \frac{k_1 l}{k_2} + \frac{(k_1^2 + k_2^2)l}{2k_2^2}[\frac{k_2\alpha}{\sqrt{l(k_1^2 + k_2^2)}}\tau + \cos(\frac{k_2\alpha}{\sqrt{l(k_1^2 + k_2^2)}}\tau + \arcsin(\frac{k_1^2 - k_2^2}{k_1^2 + k_2^2}))], \tag{3.15}$$



(a) (b)

Figure 2: Rays of (a) plane waves propagating from $x = -\infty$ and (b) circular waves generated by a point source at the origin, moving toward a shoreline. The bottom function is chosen to be that in (3.8) with $\alpha = 1$ and $l = 2$. Other parameters are: (a) $k_1 = k_2 = 1, y_0 = -1, 0, 1, 2$; (b) $k = 1, \phi = \frac{n\pi}{8}(n = 0, ..., 15)$.

for $\tau \leq \tau_M = \arccos(\frac{2k_1k_2}{k_1^2+k_2^2})$. Here τ_M is the time when rays hit the shoreline and bounce back.

At the initial stage of traveling along the beach, rays propagate in an apparently analogous way as that in figure 2a, and tend to be perpendicular to the shore. At some finite time they reach the beach where the water depth is zero. Therefore current theory breaks down in the vicinity of the shore and a more careful examination is needed to correctly describe the process taking place therein. Nevertheless, this difficulty does not hinder us from obtaining a solution to the Eikonal equation if we assign to S on $x = l$ the value carried by incoming rays and use them as the boundary conditions for reflected rays. Both incoming rays and reflected rays are plotted in figure 3.

4 Solutions to vc-cy-KdV equation.

The corresponding KdV-like equation in the cylindrical coordinate system can be obtained by a similar approach (Shen&Shen[9]); namely,

$$\eta_{1t} + (2\omega^{-1}rJ)^{-1}\frac{d\omega^{-1}rJ}{dt} + \frac{3\omega\eta_1\eta_{1\xi}}{2h} + \frac{1}{6}\omega^3 h\eta_{1\xi\xi\xi} = 0, \tag{4.1}$$

where r stands for the radial component; η_1, ω, J and h are analogously defined in the cylindrical coordinate system.

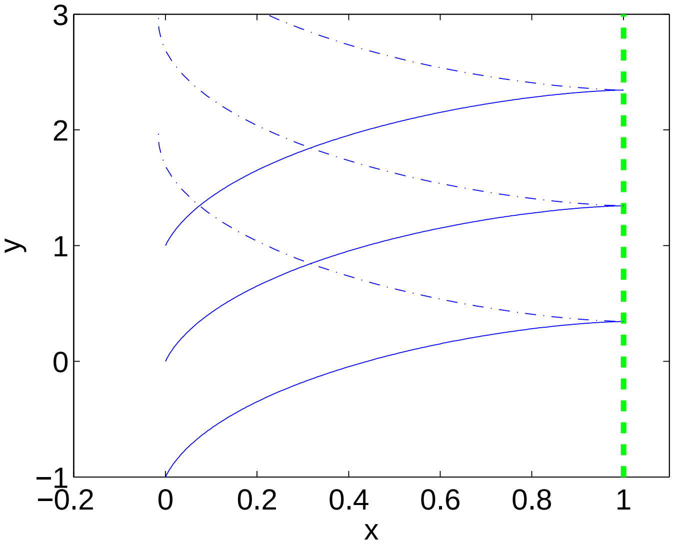


Figure 3: Rays of plane waves (solid curves) traveling over a beach with constant slope are reflected by the beach (dot-dashed curves), with $\alpha = 1, l = 1, k_1 = 1, k_2 = 8$, and $y_0 = -1, 0, 1$.

Assume now that h is a function of r only and all other variables do not depend on the azimuthal variable ϕ, then the phase function is

$$S(r, \phi, t) = -t + \int_{r_0}^{r} h^{-\frac{1}{2}} dr, \tag{4.2}$$

for some fixed radius r_0. If we introduce a function $A(t, r, \xi)$ such that $\eta_1 = Ah(r)$, then (4.1) can be written as a vc-cl-KdV equation

$$A_r + (\frac{3}{2}h^{-\frac{1}{2}})AA_\xi + (\frac{1}{6}h^{\frac{1}{2}})A_{\xi\xi\xi} + (\frac{5}{4}h_r h^{-1})A + \frac{A}{2r} = 0, \tag{4.3}$$

Equation (4.3) is the counterpart of (2.27) in the cylindrical coordinate system under the assumption of azimuthal independence.

By applying a change of variables

$$A(r, \xi) = \frac{2\sqrt{6}}{3} h^{-5/4} r^{-1/2} B(\sigma, \xi), \quad \text{and} \quad \sigma = \frac{1}{6}\int^{r} h^{1/2} dr, \tag{4.4}$$

equation (4.3) becomes

$$B_\sigma + f(\sigma)BB_\xi + B_{\xi\xi\xi} = 0, \tag{4.5}$$

where

$$f(\sigma) = 6h^{-9/4}(\int^{\sigma} h^{-1/2} d\sigma)^{-1/2} = 6\sqrt{6} r^{-1/2} h(r)^{-9/4}. \tag{4.6}$$

Equation (4.5) is in the form of (2.27), except that the "time" variable σ now has a different scale from the actual time variable t.

4.1 Exact solutions

To look for exact solutions to (4.5), we apply the following change of variables suggested by Grimshaw (appendix in [3])

$$B(\sigma,\xi) = f(\sigma)[Q(s,x) - m_1\xi], \quad x = f(\sigma)\xi, \quad s = g(\sigma), \tag{4.7}$$

where m_1 is some constant, $g(\sigma)$ is to be determined, and $Q(s,x)$ satisfies the standard KdV equation

$$Q_s + QQ_x + Q_{xxx} = 0. \tag{4.8}$$

Substituting (4.7) into (4.5) yields

$$\begin{aligned}(f' - m_1 f^3)Q + (\xi f f' - m_1\xi f^4)Q_x + (m_1^2\xi f^3 - m_1\xi f')\\ + f g' Q_s + f^4 QQ_x + f^4 Q_{xxx} = 0.\end{aligned} \tag{4.9}$$

Therefore $Q(s,x)$ will satisfy (4.8) if each bracketed term in (4.9) vanishes and $fg' = f^4$. This means that when $m_1 \neq 0$, f and g have to satisfy

$$f(\sigma) = (-2m_1\sigma + m_2)^{-1/2}, \quad g(\sigma) = \frac{1}{m_1}(-2m_1\sigma + m_2)^{-1/2}. \tag{4.10}$$

where m_2 is an integrating constant. This is consistent with the integrability condition (2.28) with $c_2 = 0$. According to (4.6), (4.10) and definition of σ, the bottom function has to satisfy

$$h(r) = (-72m_1)^{1/4}(m_3 r^{-8/9} + 1)^{1/4}. \tag{4.11}$$

When h is chosen as in (4.11), we may combine (4.4), (4.7) and any exact solution to the standard KdV to obtain a new solution to the vc-cy-KdV equation. Nevertheless, the presence of the term "$m_1\xi$" in transformation (4.7) changes the condition of the solution at infinity both in temporal and spacial variables. Consequently, solutions to the standard KdV with physical meanings become unbounded solutions to vc-cy-KdV. As an illustration to it, let us take a soliton solution to (4.8)

$$Q(s,x) = 3c\,\text{sech}^2[\frac{\sqrt{c}}{2}(x - cs - b)], \tag{4.12}$$

where b is any constant and c encodes information of amplitude, wavenumber and frequency of the soliton. Then

$$A(r,t) = \kappa_1 r^{-1}\,\text{sech}^2[\kappa_2 r^{-\frac{1}{2}}(r - \kappa_3 t) + \kappa_4 r^{-\frac{1}{2}} + \kappa_5] + \kappa_6 r^{-\frac{1}{2}}(r - \kappa_3 t), \tag{4.13}$$

for some constants $\kappa_i, (i = 1, ..., 6)$. Solution (4.13) is plotted in figure 4.

We notice that when $f(\sigma)$ is a constant ($m_1 = 0$), there is no need of applying another transformation to (4.5); thereupon any bounded solution to it remains bounded to vc-cy-KdV. In particular, solitary waves will keep their soliton-like shape except that the amplitude may vary due to combined effects of geometrical spreading and shoaling. Suppose $f = 6\lambda_1$ for some constant λ_1, then one soliton solution to (4.5) is

$$B = \frac{c}{2\lambda_1}\,\text{sech}^2[\frac{\sqrt{c}}{2}(\xi - c\sigma - b)], \quad \text{for a constant } b. \tag{4.14}$$

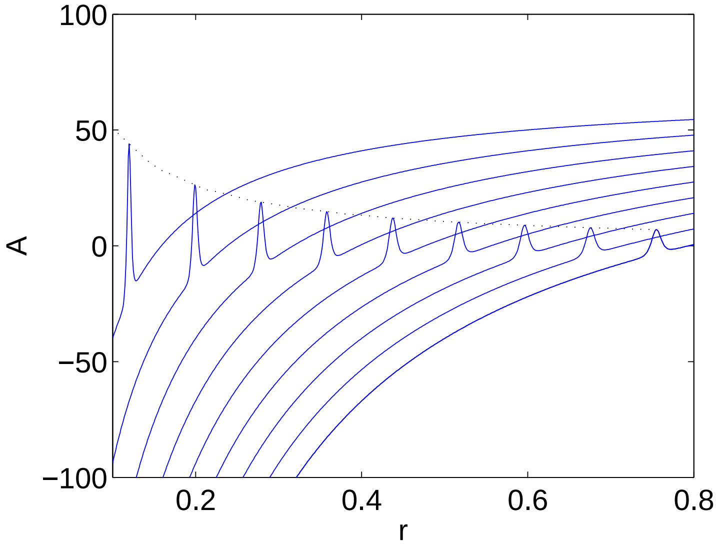


Figure 4: An unbounded solution (4.13) at different times. The dotted curve is the envelope of local solitons.

Now

$$\sigma = 2^{-\frac{35}{9}} 3^{\frac{10}{9}} \lambda_1^{-\frac{2}{9}} r^{\frac{8}{9}} - \lambda_2, \tag{4.15a}$$

$$h(\sigma) = [\frac{8\lambda_1^2}{9}(\sigma + \lambda_2)]^{-\frac{1}{4}} = (\frac{36}{\lambda_1^4})^{1/9} r^{-\frac{2}{9}}. \tag{4.15b}$$

where λ_2 is an arbitrary constant. Combining (4.4),(4.14),(4.15) and definition of ξ we obtain a bounded soliton solution to vc-cy-KdV (4.3):

$$A(r,\xi) = 2^{\frac{2}{9}} 3^{-\frac{7}{9}} r^{-\frac{2}{9}} \lambda_1^{-1} \operatorname{sech}^2[\frac{\sqrt{c}}{2}(\epsilon^{-1}\xi - c\sigma - b)], \tag{4.16}$$

which is plotted in figure 5 for $r \geq r_0$. An examination of solution (4.16) shows that for large r the amplitude decays to 0 as $r^{-2/9}$. Moreover, the dominant term inside the hyperbolic secant function is $C_1 r^{\frac{10}{9}} - C_2 t$ for some constants C_1 and C_2, indicating that the traveling speed of the soliton decays to 0 as $r^{-\frac{1}{9}}$.

4.2 Approximate solutions – Cnoidal waves.

As we can see in previous section, the choice of the topography is fairly restrictive in order for us to obtain explicit solutions. Even in the most special case when the bottom is flat, we are not able to obtain a solution which is physically meaningful. Therefore developing an approach to find approximate solutions is of necessity. We look for an approximate solution to vc-KdV (4.5) which is periodic (cnoidal waves) in this section, and non-periodic (solitary waves) in next section.

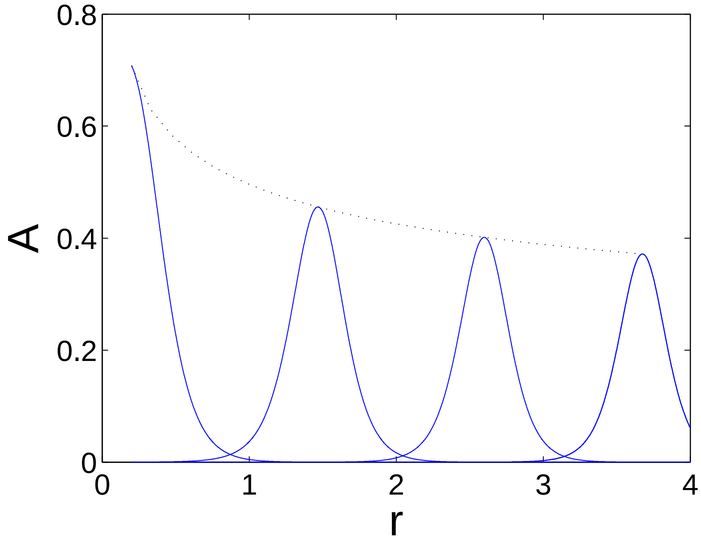


Figure 5: An exact solution $A(r, \xi)$ given by (4.16), plotted at time $t = 0, 1, 2, 3$ with parameters $c = 0.01, b = 1, \lambda_1 = 1, \lambda_2 = 0, r_0 = 0.2$ and $\epsilon = 0.01$. The dotted curve is the envelope of the moving soliton.

Assume that $f(\sigma)$ slowly varies in σ, then we can introduce a slow-time variable $T = \epsilon\sigma$ $(\epsilon << 1)$ and write $f(\sigma)$ as

$$f = f(T). \tag{4.17}$$

Denote

$$\zeta = k(\xi - \epsilon^{-1} \int^T V(T)dT), \tag{4.18}$$

where k is some constant wavenumber. We seek for a periodic solution B which is periodic in ζ with a period 2π and has the asymptotic expansion

$$B = B_0(\zeta, T) + \epsilon B_1(\zeta, T) + O(\epsilon^2). \tag{4.19}$$

Plugging it into (4.5) we obtain the leading order term as

$$-VB_{0\zeta} + fB_0B_{0\zeta} + k^2 B_{0\zeta\zeta\zeta} = 0. \tag{4.20}$$

Equation (4.20) is an ordinary differential equation in ζ, with T as a parameter; furthermore, it has a cnoidal wave solution

$$B_0(\zeta, T) = a[b + \mathrm{cn}^2(\gamma\zeta; m)] + d, \tag{4.21}$$

where

$$b = \frac{1-m}{m} - \frac{E(m)}{mK(m)}, \tag{4.22a}$$

$$a = \frac{12mK^2(m)k^2}{\pi^2 f}, \tag{4.22b}$$

$$\text{and } V = [d + a(\frac{2-m}{3m} - \frac{E(m)}{mK(m)})]f. \tag{4.22c}$$

Here notations are: $\mathrm{cn}(x;m)$ is the Jacobian elliptic function with parameter $m \in (0,1)$; $K(m)$ and $E(m)$ (or K and E for necessary abbreviations) are complete elliptic integrals of the first and second kind respectively. Since $\mathrm{cn}^2(x;m)$ is known to have a period of $2K(m)$, $\gamma = \frac{K(m)}{\pi}$. The constant b is so chosen that the integration of the sum in the bracket of (4.21) over one period is zero.

There are three free parameters in the cnoidal wave solution, which are chosen by us to be the amplitude a, mean level d and elliptic parameter m. They are functions of T (or r) describing how waves are modulated. To determine a, d and m, we apply the so-called Whitham averaging method ([11]), which considers the following two conservation laws directly deduced from (4.5)

$$\frac{\partial}{\partial t}\int_0^{2\pi} B d\zeta = 0, \tag{4.23}$$

$$\frac{\partial}{\partial t}\int_0^{2\pi} B^2 d\zeta = 0. \tag{4.24}$$

These conservation laws hold for each term in the expansion of B. It follows from (4.23) that $d = const$. Plugging (4.21) into (4.24) and using (4.22b) we get

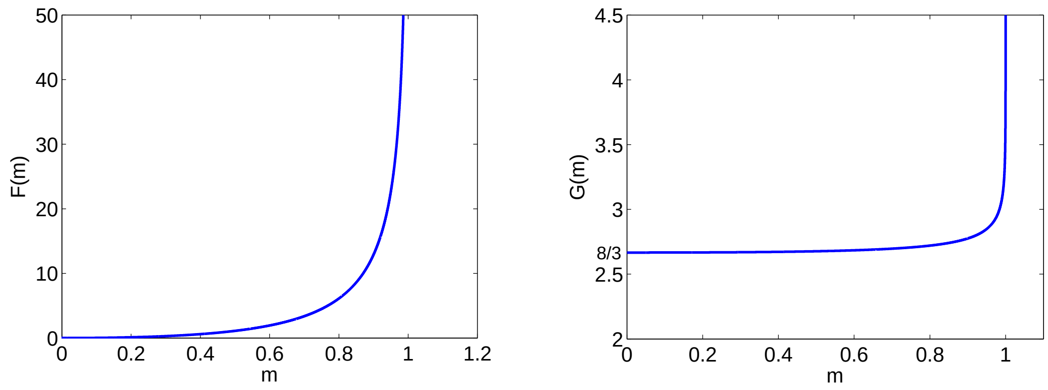
$$a^2 = \frac{m^2 K(m)^4 I_0}{F(m)} \equiv G(m) I_0, \tag{4.25}$$

$$\text{and} \quad F(m) = \frac{\pi^4 I_0}{144 k^4} f^2, \tag{4.26}$$

where

$$F(m) \equiv K(m)^2[(4-2m)E(m)K(m) - 3E(m)^2 - (1-m)K(m)^2], \tag{4.27}$$

and I_0 is a sum of a constant depending only on d and another integrating constant. $F(m)$ and $G(m)$ are plotted in figure 6. The wavenumber k and V can be determined from (4.22b) and (4.22c), respectively.



(a) (b)

Figure 6: Plots of $F(m)$ and $G(m)$ versus m.

To study the remote behavior of the amplitude of a cnoidal wave solution (4.21), us choose a representative example of bottom function

$$h(r) = (r + r_1)^{-\beta} \qquad r_1 \geq 0,\ \beta > 0. \tag{4.28}$$

Then it follows from (4.6) that

$$f(r) = 6\sqrt{6}r^{-1/2}(r + r_1)^{9\beta/4}. \tag{4.29}$$

We remark that in a more generalized case the bottom can be chosen to be $h = a_1(r + r_1)^{-\beta} + a_2$ which takes into account the slope of the basin at origin and the place where water depth vanishes.

At large r, from (4.29) we have $f \sim r^{\frac{9\beta-2}{4}}$. Thus when $\beta < \frac{2}{9}$, $f(r) \to 0$ as $r \to \infty$. From (4.26) and monotonicity of $F(m)$(figure 6a), we have $\lim\limits_{r\to\infty} m(r) = 0$. Since $\lim\limits_{m\to 0} G(m) = \dfrac{8}{3}$(figure 6b), it follows from (4.25) that the amplitude a decreases to a finite value as $r \to \infty$. Then the physical amplitude

$$\widehat{a} = h^{-5/4}r^{-1/2}a \sim r^{(5\beta-2)/4} \tag{4.30}$$

decreases to zero as well. When $\beta > \frac{2}{9}$, $\lim\limits_{r\to\infty} f(r) = \infty$, and $\lim\limits_{r\to\infty} m(r) = 1$, so $a \to \infty$. However the physical amplitude

$$\widehat{a} = h^{-\frac{5}{4}}r^{-\frac{1}{2}}a \sim F^{4(\beta-\frac{1}{3})/(9\beta-2)}, \quad \text{as } r \to \infty. \tag{4.31}$$

Therefore if $\frac{2}{9} < \beta < \frac{1}{3}$, the amplitude of cnoidal wave $\widehat{a}$ still decreases to 0. But if $\beta > \frac{1}{3}$, (4.31) still holds by the same argument, and $\widehat{a} \to \infty$ as $r \to \infty$. At the critical slope $\beta = \frac{1}{3}$, the wave amplitude remains finite and converges to a non-zero constant. It has the physical implication that the effects of geometrical spreading and shoaling are offset.

4.3 Approximate solutions – Solitary waves.

Heuristically, cnoidal waves will become solitary waves when the elliptic parameter equals unity. However, techniques discussed in section 4.2 can not be directly applied to find solitary wave solution by merely taking the limit m approaching 1, because the two processes $m \to 1$ and $\epsilon \to 0$ do not commute. A new concept of slowly-varying is required and we adopt the one given by Grimshaw[4]. Equation (4.17) is to be used as before but (4.18) is now replaced by

$$\delta = \xi - \epsilon^{-1}\int^{T} V(T)dT, \tag{4.32}$$

and (4.19) replaced by

$$B = B_0(\delta, T) + \epsilon B_1(\delta, T) + O(\epsilon^2). \tag{4.33}$$

The solution B is no longer required to be periodic in δ. Substitution of (4.33) into (4.5) gives the leading order term as

$$-VB_{0\delta} + fB_0B_{0\delta} + B_{0\delta\delta\delta} = 0, \tag{4.34}$$

which has a soliton solution

$$B_0 = a\,\mathrm{sech}^2(q\delta), \tag{4.35}$$

where $V = \frac{af}{3} = 4q^2$, for some constant q. Substituting (4.35) into the conservation law

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} B_0^2 \, d\delta = 0 \tag{4.36}$$

leads to

$$a^3 = \mathrm{const.} f. \tag{4.37}$$

Combining (4.29) and (4.37) we can see that for large r,

$$a \sim r^{(9\beta-2)/4}.$$

The physical amplitude is then

$$\widehat{a} \sim h^{-5/4} r^{-1/2} a \sim r^{2(\beta-\frac{1}{3})}, \quad \text{as} \;\; r \to \infty. \tag{4.38}$$

Therefore $\beta = \frac{1}{3}$ is again a critical value for the behavior of the amplitude.

Let us now examine when the approximate solutions of cnoidal waves and solitary waves are valid. We treat f and a, m etc. as functions of r instead of T(or σ) since the variable r represents the radial distance from the origin. Approximate solutions are only true for a slowly varying function f, which is determined by the bottom function $h(r)$ as in (4.6). Moreover, generation of circular waves (such as dropping a stone into a pond) usually involves very complicated mechanism near the origin where current theory fails. These two difficulties can be circumvented by considering the flow domain to be the whole plane excluding vicinity of the origin.

As a summary to this section, we characterize as follows the amplitude $\widehat{a}$ to vc-cy-KdV equation (4.3) in terms of β while the bottom function is in the form of $h(r) = (r + r_1)^{-\beta}$:

(i) when $0 < \beta < \frac{1}{3}$, $\widehat{a} \to \infty$ as $r \to \infty$;

(ii) when $\beta = \frac{1}{3}$, $\widehat{a} \to$ const. as $r \to \infty$;

(iii) when $\frac{1}{3} < \beta < 2$, $\widehat{a} \to 0$ as $r \to \infty$.

It is perhaps an interesting but not a surprising fact that both cnoidal waves and solitary waves share the same critical value $\beta = \frac{1}{3}$, even though the analysis leading to the result are not identical.

5 Discussion.

Equation (2.27) was originally derived by Shen&Keller[8] for a compressible fluid rotating at a constant angular velocity about the $z-$axis. In that case an adiabatic equation must be added to the governing equations and the equation of motion must be modified to include the effect of rotation. Therefore our equation is a special case of the KdV equation obtained by them. The features of nonlinearity and dispersion of waves traveling in shallow water are captured in this variable coefficient KdV equation.

The critical value $\beta = \frac{1}{3}$ for which wave amplitude approaches to zero/finity/infinity may not be verified in practice, since the occurrence of reflected waves from the rising topography is neglected. As remarked in section 3.3, waves are certainly reflected if $\beta < 2$. This raises the possibility that returning waves interact with advancing waves and modify the wave amplitude in a non-trivial way. Mathematically, a possible remedy is to allow the sign in front of variable t in the phase function (eqn.(4.2)) to be reversed, giving rise to another vc-cy-KdV equation. The new KdV equation needs to be solved together with the existing one.

Acknowledgement

I am grateful to Joseph Keller for his advice during the whole summer. His insightful ideas always enlightened me to view and solve difficult problems from broader perspectives. My acknowledgement is also to Harvey Segur and Roger Grimshaw for their principle lectures as well as many helpful discussions afterwards. Finally I would like to extend my thanks to all GFD staff and fellows for making this summer program so exciting and memorable.

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