1 Introduction


The theory of upper bounds for functionals of turbulent flows provides rigorous bounds for transport properties. It also indicates characteristic properties of extremalizing vector fields, which are reflected in observations of turbulent flows and thus can provide some insights into properties of turbulence.

2 Upper Bounds on Momentum Transport Between Two Moving Parallel Plates

In this section, we consider a flow between two moving parallel plates as shown in Fig. 1. Using the distance $d$ between two plates as length scale, and $d^2/\mu$ as time scale, we write the Navier-Stokes equation for the incompressible fluid in the form

$$\frac{\partial}{\partial t} \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + 2\Omega \times \mathbf{v} = -\nabla p + \nabla^2 \mathbf{v}$$

(1)

$$\nabla \cdot \mathbf{v} = 0.$$  

(2)

We separate the velocity field $\mathbf{v}$ into a mean and a fluctuating part:

$$\mathbf{v} = \mathbf{U} + \mathbf{\hat{v}} \quad \text{with} \quad \mathbf{\bar{v}} = 0, \quad \nabla \equiv \mathbf{U}(z, t)$$  

(3)

where
\[
\cdots \equiv \lim_{L \to \infty} \frac{1}{4L^2} \int_{-L}^{L} \int_{-L}^{L} \cdots dx. \tag{4}
\]

We also separate the fluctuating part of the velocity field \( \mathbf{\hat{v}} \) into components perpendicular and parallel to the plates as

\[
\mathbf{\hat{v}} \equiv \mathbf{\hat{u}} + \mathbf{k} \hat{w} \text{ with } \mathbf{\hat{u}} \cdot \mathbf{k} = 0. \tag{5}
\]

Figure 1: Schematic sketch of a flow between two moving parallel plates.

For \( \mathbf{\Omega} \cdot \mathbf{k} = 0 \) (e.g. Taylor-Couette case), since \( \mathbf{U} \) does not have a z-component because of the continuity equation, the average over planes \( z=\text{constant} \) of (1) yields

\[
\frac{\partial}{\partial t} \mathbf{U} + \mathbf{\bar{v}} \cdot \nabla \mathbf{\bar{u}} = \frac{\partial^2}{\partial z^2} \mathbf{U} \tag{6}
\]

\[
\mathbf{\bar{v}} \cdot \nabla \mathbf{\bar{w}} = -\frac{\partial}{\partial z} \mathbf{\bar{p}} - 2 \mathbf{\Omega} \times \mathbf{U}. \tag{7}
\]

Subtracting (6) and (7) from the corresponding components of (1), we obtain the following equation for the fluctuating velocity field \( \mathbf{\hat{v}} \):

\[
\frac{\partial}{\partial t} \mathbf{\hat{v}} + \mathbf{\hat{v}} \cdot \nabla \mathbf{\hat{v}} - \mathbf{\bar{v}} \cdot \nabla \mathbf{\bar{v}} + \mathbf{U} \cdot \nabla \mathbf{\hat{v}} + \mathbf{\hat{v}} \cdot \nabla \mathbf{U} + 2 \mathbf{\Omega} \times \mathbf{\hat{v}} = -\nabla \mathbf{\bar{p}} + \nabla^2 \mathbf{\hat{v}}. \tag{8}
\]

After multiplying the above equation with \( \mathbf{\hat{v}} \), taking the average over the entire fluid layer, and using the boundary conditions that \( \mathbf{\hat{v}} \) vanishes at \( z = \pm \frac{1}{2} \), we have the energy relationship

\[
\frac{1}{2} \frac{d}{dt} \langle |\mathbf{\hat{v}}|^2 \rangle + \langle |\nabla \mathbf{\hat{v}}|^2 \rangle + \langle \mathbf{\hat{u}} \cdot (\hat{w} \frac{\partial}{\partial z}) \mathbf{U} \rangle = 0 \tag{9}
\]

where

\[
\langle \cdots \rangle = \int_{-\frac{1}{2}}^{\frac{1}{2}} \cdots dz. \tag{10}
\]

The above energy relationship (9) can be further simplified if we restrict our attention to the fluid flow under stationary conditions:

\[
\frac{\partial}{\partial t} \mathbf{U} = 0; \quad \frac{d}{dt} \langle |\mathbf{\hat{v}}|^2 \rangle = 0. \tag{11}
\]
The equation (6) under above condition yields
\[
\frac{d}{dz} U = \overline{w\hat{u}} - \langle |\hat{w}\hat{u}| \rangle - Re \cdot i.
\] (12)

Hence, using the above equation, we obtain the final form of the energy balance
\[
\langle |\nabla \hat{v}|^2 \rangle + \langle \hat{u}\hat{w} \cdot (\overline{\hat{u}\hat{w}} - \langle \hat{u}\hat{w} \rangle) \rangle - Re \langle \hat{u}_x\hat{w} \rangle = 0.
\] (13)

Here, the identity
\[
\langle \overline{\hat{u}\hat{w}}^2 \rangle - \langle \hat{u}\hat{w} \rangle^2 = \langle |\overline{\hat{u}\hat{w}} - \langle \hat{u}\hat{w} \rangle|^2 \rangle
\] (14)

has been used.

The momentum transport between two moving plates is obtained from its value at the boundary
\[
M \equiv -\frac{\partial U}{\partial z} \bigg|_{z=\frac{1}{2}} = \langle \hat{w}\hat{u}_x \rangle + Re.
\] (15)

Since \( \langle \hat{w}\hat{u}_x \rangle \geq 0 \), the momentum transport is bounded from below by the value of the laminar solution and increases by \( \langle \hat{w}\hat{u}_x \rangle \) for turbulent flow. Thus, the goal here is to derive an upper bound for \( \langle \hat{u}_x\hat{w} \rangle \) at a given value of \( Re \) and this leads us to the formulation of the following variational problem. For a given \( \mu \), find the minimum \( R(\mu) \) of the functional
\[
R(\mathbf{v}, \mu) \equiv \frac{\| \nabla \mathbf{v} \|^2}{\langle u_x w \rangle} + \mu \frac{\| \overline{\mathbf{u}w} - \langle \mathbf{u}w \rangle \|^2}{\langle u_x w \rangle^2}
\] (16)

among all vector fields with \( \mathbf{v} = 0 \) at \( z = \pm \frac{1}{2} \) where
\[
\mathbf{v} \equiv \mathbf{u} + \mathbf{k}w, \ \mathbf{u} \cdot \mathbf{k} \equiv 0.
\] (17)

Thus, the Euler-Lagrange equations for a stationary value of \( R(\mathbf{v}, \mu) \) are
\[
\nabla^2 \mathbf{v} - \nabla \pi = w \frac{d}{dz} U^* + \mathbf{k}u \cdot \frac{d}{dz} U^*
\] (18)

where
\[
\frac{d}{dz} U^* = \overline{\mathbf{w}\mathbf{u}} - \langle \mathbf{w}\mathbf{u} \rangle - \left( R - \frac{\| \nabla \mathbf{v} \|^2}{2\langle u_x w \rangle} \right) i
\] (19)

Since the functional is homogeneous, the normalization \( \langle \hat{u}_x\hat{w} \rangle = \mu \) can be assumed.

The proof for \( \frac{dR(\mu)}{d\mu} = \frac{(\overline{\mathbf{w}\mathbf{u}} - \langle \mathbf{w}\mathbf{u} \rangle)^2}{\langle u_x w \rangle^2} \) is as follows:
\[
(\mu^* - \mu') \frac{(\overline{\mathbf{w}^*\mathbf{u}^*} - \langle \mathbf{w}^*\mathbf{u}^* \rangle)^2}{\langle \mathbf{w}^* u_x^* \rangle^2} = R(\mathbf{v}^*, \mu^*) - R(\mathbf{v}^*, \mu') \leq R(\mu^*) - R(\mu') \leq R(\mathbf{v}', \mu^*) - R(\mathbf{v}', \mu') \leq (\mu^* - \mu') \frac{(\overline{\mathbf{w}'\mathbf{u}'} - \langle \mathbf{w}'\mathbf{u}' \rangle)^2}{\langle \mathbf{w}' u_x' \rangle^2}
\]

where \( \mathbf{v}^* \) and \( \mathbf{v}' \) are the extremalizing vector fields for \( \mu^* \) and \( \mu' \), respectively. For \( \mu^* \to \mu' \), the above result follows.
3 Upper Bounds on the Heat Transport in a Porous Layer

In this section, we consider a thermal convection in a porous medium as shown in Fig. 2. Using the distance $d$ between two plates as length scale, $d^2/\kappa$ as time scale, $\kappa/d$ as velocity scale, and $(T_2-T_1)/R$ as temperature scale, we write dimensionless equations based on Darcy-Law as

\[-u + kT - \nabla p = B(\frac{\partial}{\partial t} u + u \cdot \nabla u) \approx 0\]  \hspace{1cm} (20)

\[\nabla \cdot u = 0\] \hspace{1cm} (21)

\[\nabla^2 T = (\frac{\partial}{\partial t} + u \cdot \nabla) T\] \hspace{1cm} (22)

where

\[B \equiv \frac{\kappa K}{d^2 \nu}\] \hspace{1cm} (23)

\[R \equiv \frac{\gamma g K d (T_2 - T_1)}{\nu \kappa} \] \hspace{1cm} (24)

and $K$ is the Darcy permeability coefficient.

We separate the temperature field $T$ into a mean and a fluctuating part

\[T = \overline{T} + \theta \text{ , with } \overline{\theta} = 0\] \hspace{1cm} (25)

By subtracting the horizontal average of (22) from (22), we obtain

\[\frac{\partial}{\partial t} \overline{T} + u \cdot \nabla \theta = \frac{\partial^2}{\partial z^2} \overline{T}\] \hspace{1cm} (26)

\[(\frac{\partial}{\partial t} + u \cdot \nabla \theta) + w \frac{\partial T}{\partial z} - \frac{\partial}{\partial z} w \theta = \nabla^2 \theta\] \hspace{1cm} (27)

Assuming the statistically stationary turbulence, we integrate (26) and obtain
\[
\frac{\partial}{\partial z} T = \overline{w\theta} - \langle w\theta \rangle - R
\]

(28)

By multiplying (20) by \( u \) and (27) by \( \theta \), taking the average over the whole porous layer, and using (28), we obtain two dissipation integral relationships:

\[
\langle |u|^2 \rangle = \langle w\theta \rangle
\]

(29)

\[
\langle |\nabla \theta|^2 \rangle + \langle \overline{w\theta - w\theta} \rangle^2 = R \langle w\theta \rangle.
\]

(30)

The dimensionless heat transport across the porous layer can be obtained from its value at the boundary:

\[
H = \frac{\partial T}{\partial z} \bigg|_{z = \pm \frac{1}{2}} = R + \langle w\theta \rangle \geq R.
\]

(31)

Since \( \langle w\theta \rangle \) is always positive from (29), the heat transport for the turbulent flow is always greater than that for the laminar flow, and it is bounded from below by the value of the laminar solution.

The goal here is to find an upper bound on the heat transport or \( \langle w\theta \rangle \) at a given value of \( R \). We are thus led to the formulation of the following variational problem. For given \( \mu > 0 \), find the minimum \( P(\mu) \) of the functional

\[
P(\mathbf{u}, \theta, \mu) \equiv \frac{\langle |u|^2 \rangle \langle |\nabla \theta|^2 \rangle + \mu \langle |\overline{w\theta - w\theta}|^2 \rangle}{\langle w\theta \rangle^2}
\]

(32)

for all fields \( \mathbf{u} \) and \( \theta \), which satisfy the constraint \( \nabla \cdot \mathbf{u} = 0 \) and the boundary condition \( w = \theta = 0 \) at \( z = \pm \frac{1}{2} \). First, from the general form of the dissipation integral

\[
\langle |u|^2 \rangle \equiv \langle \nabla^2 v \Delta_2 v \rangle + \langle |k \times \nabla \psi|^2 \rangle
\]

(33)

and the property

\[
w = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)v \equiv -\Delta_2 v
\]

(34)

it is clear that the minimum of the functional is obtained for \( \nabla \times k \psi = 0 \). Hence, the variational problem now depends only on the scalar variables \( v \) and \( \theta \).

The Euler-Lagrange equations for a stationary value can be thus written as

\[
\langle |\nabla \theta|^2 \rangle \nabla^2 w - [P\langle w\theta \rangle + \mu(\langle w\theta \rangle - \overline{w\theta})] \Delta_2 \theta = 0
\]

(35)

\[
\langle \nabla^2 v \Delta_2 v \rangle \nabla^2 \theta + [P\langle w\theta \rangle + \mu(\langle w\theta \rangle - \overline{w\theta})] w = 0
\]

(36)
Now, the nonlinearity is only through $z$-dependence and the equations are linear with respect to the $x, y$ dependence. This property allows us to write solutions in the form of superposition of waves. Because of the homogenity with respect to $x$ and $y$ in $w$ and $\theta$, we can impose the following normalization conditions:

$$\langle |\nabla \theta|^2 \rangle = 1 \quad (37)$$

$$\langle \nabla^2 v \Delta_2 v \rangle = \langle |k \times \nabla v|^2 \rangle = 1 \quad (38)$$

Then, we introduce the following general solutions for $w$ and $\theta$

$$w = w^{(N)} = \sum_{k=1}^{N} \frac{1}{2} \alpha_n^2 w_n(z) \Phi_n(x, y) \quad (39)$$

$$\theta = \theta^{(N)} = \sum_{k=1}^{N} \alpha_n^{-1/2} \theta_n(z) \Phi_n(x, y) \quad (40)$$

where $\Phi_n$ satisfies the equation:

$$\Delta_2 \Phi_n = -\alpha_n^2 \Phi_n \quad (41)$$

and the orthonormalization condition

$$\overline{\Phi_n \Phi_m} = \delta_{mn} \quad (42)$$

Then, the Euler-Lagrange equations can be reduced to

$$\left( \frac{\partial^2}{\partial z^2} - \alpha_n^2 \right) w_n + \alpha_n \Psi \theta_n = 0 \quad (43)$$

$$\left( \frac{\partial^2}{\partial z^2} - \alpha_n^2 \right) \theta_n + \alpha_n \Psi w_n = 0 \quad (44)$$

where

$$\Psi \equiv P \sum_{n=1}^{N} \langle w_n \theta_n \rangle + \mu \sum_{n=1}^{N} \left( \langle w_n \theta_n \rangle - \overline{w_n \theta_n} \right. \quad (45)$$

The above equations have the following properties [8]:

1. By considering the equations for $w_n + \theta_n$ and $w_n - \theta_n$, we can obtain

$$w_n = \theta_n \quad (46)$$
Thus, the problem can be reduced to

\[
\left( \frac{\partial^2}{\partial z^2} - \alpha_n^2 \right) \theta_n + \alpha_n \Psi \theta_n = 0 \quad (47)
\]

(2) The functions \( \theta_n(z) \) are either symmetric or antisymmetric in \( z \).

(3) Since \( \theta_n \equiv \theta_m \) follows from \( \alpha_n = \alpha_m \), it can be assumed that all \( \alpha_n \) are different.

(4) For \( m \neq n \), by subtracting the \( n \)-th equation of (47) multiplied by \( \alpha_n^{-1} \theta_m \) from the \( m \)-th equation multiplied by \( \alpha_m^{-1} \theta_n \), and averaging it using the partial integration, we obtain an important property

\[
\langle \theta'_m \theta'_n \rangle - \alpha_m \alpha_n \langle \theta_m \theta_n \rangle = 0 \quad (48)
\]

where \( \theta'_m \) denotes the \( z \)-derivative of \( \theta_m \).

(5) Minimization of \( P(\theta_n, \alpha_n, \mu) \) with respect to \( \alpha_n \) yields

\[
\frac{\partial}{\partial \alpha_n} I = 0 \quad (49)
\]

\[
\alpha_n^2 = \frac{\langle \theta'_n \theta'_m \rangle}{\langle \theta_n \theta_m \rangle} \quad (50)
\]

References


References


