Forced Non-normal Convection

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Abstract

We consider non-linear amplitude equations for double and triple diffusive convection close to marginal stability. We study the effect of non-normality on double and triple diffusive convection. We also consider the effect of small amplitude noise on the system. We observe that non-normality is not sufficient to take system away from basin of attraction of stable fixed points. Small amplitude noise along with non-normality can cause the system to go away from basin of attraction of stable fixed point. We also discuss the different regimes of behavior observed as the frequency of noise is changed.

1 Introduction

The importance of non-normality in transition to turbulence in shear flows has been investigated by many researchers. Non-normality was proposed as a possible explanation (c.f. Trefethen *et al.* (1993), Grossmann (2000)) for turbulence observed at *Re* numbers lower than those predicted by linear stability theory for shear flows. Non-normality is not important for Rayleigh convection as the linear operator is normal. We study the effect of non-normality in case of double diffusive and triple diffusive convection. Small noise can be amplified due to non-normality as observed by Gebhardt & Grossmann (1994). We also study the effect of small amplitude noise in case of double diffusive convection. In all cases we will study the amplitude equations for the system close to marginal stability.

Double diffusive convection, first suggested by Stommel *et al.* (1956) is a important geophysical phenomenon. Double diffusive convection is an important process in formation of stars. Double diffusive convection is also important in order to understand the ocean convection. It has been observed that close to marginal stability double diffusive convection can start even with small noise as shown in fig 1 from Shirtcliffe (1967).

Non-normality as a possible cause of instability was first considered by Trefethen *et al.* (1993) and Gebhardt & Grossmann (1994). Trefethen *et al.* (1993) suggested a nonlinear bootstrapping mechanism consisting of amplification of disturbances due to non-normality, resulting in nonlinear mixing such that output of nonlinear mixing is again amplified by non-normality. This mechanism is shown in fig 2. A more detailed description of works to study non-normality as possible pathway to turbulence is given in review paper by Grossmann (2000).

Most of the literature concerning study of non-normality is confined to their study for shear flows such as Baggett *et al.* (1993). The linear operator for shear flows are highly

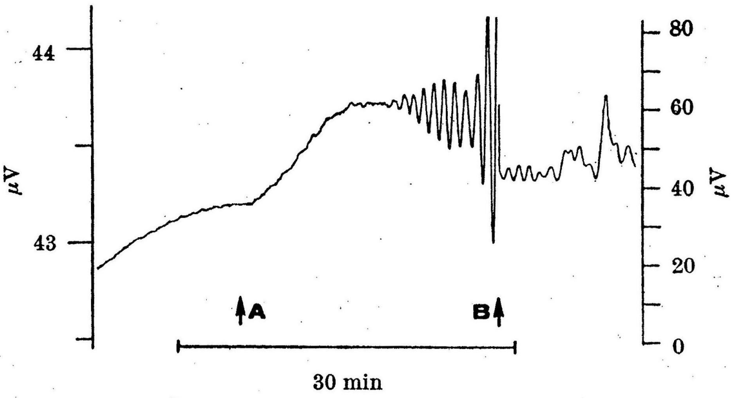

Figure 1: A section of temperature record near marginal stability. Differential output of two thermocouples is plotted against time. This transition is possible below marginal stability with small noise due to non-normality. Plot from Shirtcliffe (1967)

non-normal. The linear operator for the nonlinear amplitude equation obtained for double and triple diffusive convection obtained by Arneodo *et al.* (1985) and also Coullet & Spiegel (1983) is non-normal too. This suggests the possibility that non-normality can play important role even for convection flows.

Role of noise in non-linear dynamical systems has been studied for long time, Locher *et al.* (1998) is a review article listing important works done in the field. Even though we confine our study to single frequency small amplitude noise, our observations are in close correlation to observation for stochastic noise in dynamical systems. We find our results support the works done by Ganopolski & Rahmstorf (2002), where they suggest the possibility of climate change due to stochastic resonance.

In §2 we describe the system of equation considered. Linear analysis is described in §3. We obtain linear amplitude equations for our system in §4.We obtain nonlinear terms for the amplitude equations in §5. Finally the results are described in §6.

2 Equations

We consider two dimensional motion in a box of depth d (in z), width $\pi d/a$ (in x) and infinite length (in y), c.f. fig (3). We choose $\Delta\Theta$ and $\Delta\Sigma$, the magnitudes of the vertically impressed differences of temperature and salinity, as the units of temperature and salinity. The length and time units are d and d^2/κ_T, where κ_T is the thermal diffusivity. The equations are, c.f. Arneodo *et al.* (1985) & Chandrasekhar (1961),

$$\partial_t \Delta\Psi = \sigma\Delta^2\Psi - \sigma R\partial_x\Theta + \sigma\tau S\partial_x\Sigma + \sigma^2 T\partial_z\Upsilon + J(\Psi, \Delta\Psi), \quad (1)$$

$$\partial_t\Theta = \Delta\Theta - \partial_x\Psi + J(\Psi, \Theta), \quad (2)$$

$$\partial_t\Sigma = \tau\Delta\Sigma + \partial_x\Psi + J(\Psi, \Sigma), \quad (3)$$

$$\partial_t\Upsilon = \sigma\Delta\Upsilon - \partial_z\Psi + J(\Psi, \Upsilon). \quad (4)$$

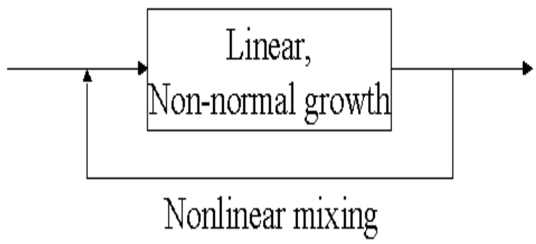

Figure 2: The suggested bootstrapping mechanism by Trefethen *et al.* (1993) leads to growth and instability.

where,
$J(f,g) = \partial_x f \partial_z g - \partial_z f \partial_x g, \quad \Delta = \partial_x^2 + \partial_z^2.$

The parameters are defined as,

$$
\begin{aligned}
R &= \frac{gd^3}{\kappa_T \nu}\left(\frac{\Delta\rho}{\rho}\right)_T, \\
S &= \frac{gd^3}{\kappa_S \nu}\left(\frac{\Delta\rho}{\rho}\right)_S, \\
T &= \left(\frac{2\Omega d^2}{\nu}\right)^2, \\
\sigma &= \frac{\nu}{\kappa_T}, \\
\tau &= \frac{\kappa_S}{\kappa_T}.
\end{aligned} \tag{5}
$$

where, R is thermal Rayleigh number, S is salinity Rayleigh number, T is Taylor number, σ is Prandtl number, τ is Lewis number, $(\Delta\rho/\rho)_T$ is the density difference solely due to imposed temperature difference and $(\Delta\rho/\rho)_S$ is density difference solely due to imposed salinity difference. Ω is the rotation rate about the z-axis.

The solution is independent of y. Ψ is the stream function for the x and z velocities, Θ and Σ are the deviations of temperature and salinity from their static values, and Υ is the y velocity divided by $\sigma T^{1/2}$.

We can express above equations concisely as

$$\partial_t \mathbf{L} U = \mathbf{M}_\lambda U + \mathbf{N}(U) \tag{6}$$

where

$$U(x,z,t) = \| \Psi, \theta, \Sigma, \Upsilon \|^T . \tag{7}$$

The parameter vector λ is

$$\lambda = (R, S, T, \sigma, \tau, a). \tag{8}$$

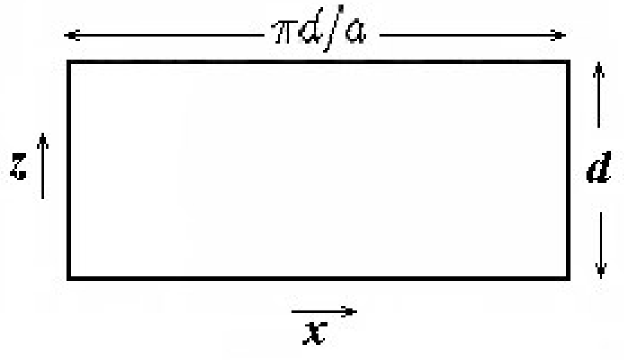

Figure 3: Two dimensional box of depth d and length $\pi d/a$.

The linear operators in eq. (6) are,

$$\mathbf{L} = \begin{vmatrix} \nabla^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \tag{9}$$

and

$$\mathbf{M}_\lambda = \begin{vmatrix} \sigma\nabla^4 & -\sigma R\partial_x & \sigma\tau S\partial_x & \sigma^2 T\partial_z \\ -\partial_x & \nabla^2 & 0 & 0 \\ \partial_x & 0 & \tau\nabla^2 & 0 \\ -\partial_z & 0 & 0 & \sigma\nabla^2 \end{vmatrix} \tag{10}$$

The non-linear term is,

$$\mathbf{N}(\mathbf{U}) = (\mathbf{L}\partial_z U E')\partial_x U - (\mathbf{L}\partial_x U E')\partial_z U \tag{11}$$

where

$$E' = \begin{vmatrix} 1 & 0 & 0 & 0 \end{vmatrix}$$

is the transpose of E.

3 Linear Theory

The linearized form of equation (6) admits solution of form,

$$U = U_{mn} * \Xi_{mn} e^{st} \tag{12}$$

where, U_{mn} is a constant four component vector. Ξ_{mn} is defined as

$$\Xi_{mn} = \begin{vmatrix} \sin(max) & \sin(n\pi z) \\ \cos(max) & \sin(n\pi z) \\ \cos(max) & \sin(n\pi z) \\ \sin(max) & \cos(n\pi z) \end{vmatrix}$$

and the operator $*$ is defined such that

$$\begin{vmatrix} a \\ b \\ c \\ d \end{vmatrix} * \begin{vmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{vmatrix} = \begin{vmatrix} a\alpha \\ b\beta \\ c\gamma \\ d\delta \end{vmatrix}$$

s is a root of,

$$\mathbf{det} \parallel M_{mn} - L_{mn} s \parallel = 0 \tag{13}$$

where M_{mn} and L_{mn} are the matrix representations of the restrictions of $\mathbf{M}$ and $\mathbf{L}$ on Ξ_{mn}. They can be found by substitution of eq (12) into eq (6).

$$M_{mn}(\lambda) = \begin{vmatrix} \sigma q_{mn}^4 & ma\sigma R & -ma\sigma\tau S & -n\pi\sigma^2 T \\ -ma & -q_{mn}^2 & 0 & 0 \\ ma & 0 & -\tau q_{mn}^2 & 0 \\ -n\pi & 0 & 0 & -\sigma q_{mn}^2 \end{vmatrix} \tag{14}$$

and

$$L_{mn}(a) = \begin{vmatrix} -q_{mn}^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \tag{15}$$

where

$$q_{mn}^2 = m^2 a^2 + n^2 \pi^2 \tag{16}$$

The fundamental in z is the most unstable of the vertical modes (as in Rayleigh- Bénard convection, Chandrasekhar (1961)), so we assume instability is encountered only for $n = 1$. When instability first occurs, it does so for a given value of a with $m = 1$ (Arneodo *et al.* (1985)). For $m = 1$ and $n = 1$, eq (13) is a quartic,

$$s^4 + P_3 s^3 + P_2 s^2 + P_1 s + P_0 = 0 \tag{17}$$

where,

$$\begin{aligned} P_0 &= \pi^2\sigma^2\tau T q^2 - a^2\sigma^2\tau S q^2 - a^2\sigma^2\tau R q^2 + \sigma^2\tau q^8, \\ P_1 &= \pi^2\sigma^2(\tau+1)T - a^2\sigma\tau(\sigma+1)S - a^2\sigma(\sigma+\tau)R + \sigma(\sigma\tau+\sigma+2\tau)q^6, \\ P_2 &= \frac{\pi^2\sigma^2 T}{q^2} - \frac{a^2\sigma\tau S}{q^2} - \frac{a^2\sigma R}{q^2} + (\sigma^2+2\sigma\tau+2\sigma+\tau)q^4, \\ P_3 &= (1+2\sigma+\tau)q^2. \end{aligned} \tag{18}$$

where $q = q_{11}$.

We are looking for triple polycritiality point. This is a point in phase space of R, S and T such that $R = R_0$, $S = S_0$ and $T = T_0$ given by eq (23) where we have three marginally stable modes and all other modes are stable. Close to this point behavior of the system can be obtained by studying the behavior of these three stable modes. The condition for this is given by, c.f. Arneodo *et al.* (1985),

$$P_0 = P_1 = P_2 = 0 \tag{19}$$

and eq (17) becomes

$$s^3(s+P_3)=0 \tag{20}$$

Dividing eq (17) by $(s+P_3)$ we get,

$$s^3+P_2s+(P_1-P_2P_3)+\frac{P_0-P_3(P_1-P_2P_3)}{s+P_3}=0 \tag{21}$$

In the neighborhood of the polycriticality condition (eq (19)), we have roots $|s| \ll P_3$ and we develop the remainder. We obtain an cubic critical polynomial for the marginally stable modes.

$$P_c(s) \equiv s^3+\mu_2 s^2+\mu_1 s+\mu_0=0 \tag{22}$$

where,

$$\mu_0=\frac{P_0}{P_3}, \qquad \mu_1=\frac{P_1}{P_3}-\frac{P_0}{P_3^2},$$
$$\mu_2=\frac{P_2}{P_3}-\frac{P_1}{P_3^2}+\frac{P_0}{P_3^3}$$

The condition for polycriticality, eq (19), can be written as,

$$\begin{aligned} R_0 &= \frac{q^6(\tau+2\sigma)}{a^2\sigma(1-\sigma)(1-\tau)}, \\ S_0 &= \frac{q^6\tau^2(1+2\sigma)}{a^2\sigma(\sigma-\tau)(1-\tau)}, \\ T_0 &= \frac{q^6\sigma(1+\sigma+\tau)}{\pi^2(\sigma-\tau)(1-\sigma)}. \end{aligned} \tag{23}$$

Let λ_0 be a point on the polycritical surface defined by eq(23) or equivalently ($\mu \equiv (\mu_0,\mu_1,\mu_2)=0$). Normal mode associated with $s=0$ satisfies,

$$\mathbf{M}_{\lambda_0}\phi * \Xi_{11}=0 \tag{24}$$

where ϕ is constant vector. Let $M_0=M_{11}(\lambda_0)$ and $L_0=L_{11}(a)$. For $s=0$ in eq(13) is det $M_0=0$, and we solve for eigenvectors and generalized eigenvectors,

$$M_0\phi=0, \quad M_0\psi=L_0\phi, \quad M_0\chi=L_0\psi. \tag{25}$$

Solving eq(25) we obtain,

$$\begin{aligned} \phi &= \| 1, \frac{-a}{q^2}, \frac{a}{\tau q^2}, \frac{-\pi}{\sigma q^2} \|^T, \\ \psi &= \| 0, \frac{a}{q^4}, \frac{-a}{\tau^2 q^4}, \frac{\pi}{\sigma^2 q^4} \|^T, \\ \chi &= \| 0, \frac{-a}{q^6}, \frac{a}{\tau^3 q^6}, \frac{-\pi}{\sigma^3 q^6} \|^T . \end{aligned} \tag{26}$$

4 Linear amplitude equation

The differential equations satisfied by coefficients (amplitudes) of the vectors are called amplitude equations. Considering the linearized form of eq(6),

$$\partial_t U = \mathbf{L^{-1}} \mathbf{M}_\lambda U \tag{27}$$

Near polycriticality, only three characteristic solutions are nearly marginal, rest are damped normal modes. At polycriticality we define U as,

$$U = [A(t)\phi + B(t)\psi + C(t)\chi] * \Xi_{11}(x, z) \tag{28}$$

We want to derive an equation for amplitude vector $A = (A, B, C)$. For $\lambda = \lambda_0$ we have,

$$\dot{A} = JA \tag{29}$$

Extending it to parameter space where $\lambda \neq \lambda_0$. Suppose we select three vectors $\phi_\lambda * \Xi_{11}$, $\psi_\lambda * \Xi_{11}$ and $\chi_\lambda * \Xi_{11}$, that together form the stable modes. So, we have

$$U = [A(t)\phi_\lambda + B(t)\psi_\lambda + C(t)\chi_\lambda] * \Xi_{11}(x, z) \tag{30}$$

Thus the eq (29) is deformed into linear amplitude equation for $\lambda \neq \lambda_0$,

$$\dot{A} = K_\lambda A \tag{31}$$

where K_λ satisfies following conditions,

$$\begin{aligned} \mathbf{K}_{\lambda_0} &= \mathbf{J}, \\ \mathbf{det}(\mathbf{K}_\lambda - s\mathbf{I}) &= P_c(s). \end{aligned} \tag{32}$$

Since there are only three free parameters in $P_c(s)$, we want to express K_λ in only three parameters. The Jordan-Arnold canonical form has this property and we use it, c.f. Arneodo *et al.* (1985). This gives us a third order equation which can be written as,

$$\dddot{A} + \mu_2 \ddot{A} + \mu_1 \dot{A} + \mu_0 A = 0 \tag{33}$$

or writing it as,

$$\begin{aligned} \dot{A} &= B, \\ \dot{B} &= C, \\ \dot{C} &= -\mu_2 C - \mu_1 B - \mu_0 A. \end{aligned} \tag{34}$$

which is equivalently to

$$\dot{A} = J_\mu A \tag{35}$$

where,

$$J_\mu = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\mu_0 & -\mu_1 & -\mu_2 \end{vmatrix} \tag{36}$$

This is Jordan form of K_λ.

5 Nonlinear terms

We seek nonlinear terms in amplitude equation eq(35) of form,

$$\dot{A} = JA + g(A) \tag{37}$$

Employing the general method used in Coullet & Spiegel (1983), we express the time dependence of $U(x,z,t)$ in terms of $\mathbf{A}$ as,

$$U(x,z,t) = V\left[x, z, \mathbf{A}(t)\right] \tag{38}$$

Substituting eq(37) and eq(38) in eq(6) we obtain

$$\mathcal{L}\mathbf{V} = \mathbf{N}(\mathbf{V}) - (\mathbf{g}\cdot\partial_{\mathbf{A}})\,\mathbf{L}\mathbf{V} \tag{39}$$

where,

$$\begin{aligned} \mathcal{L} &= \mathbf{D}\mathbf{L} - \mathbf{M}, \\ \mathbf{D} = (\mathbf{J}\mathbf{A})\cdot\partial_{\mathbf{A}} \quad &, \qquad \mathbf{M} = \mathbf{M}_{\lambda_{\mathbf{0}}}. \end{aligned}$$

We expand $\mathbf{V}$ and $\mathbf{g}$ in Taylor series in $\mathbf{A}$ and denote partial sum of all terms of degree κ as $\mathbf{V}_\kappa$ and $\mathbf{g}_\kappa$. Thus $\mathbf{V}_\kappa(x,z,c\mathbf{A}) = c^\kappa \mathbf{V}_\kappa(x,z,\mathbf{A})$. We now need to solve,

$$\mathcal{L}\mathbf{V}_\kappa = \mathcal{I}_\kappa - (\mathbf{g}_\kappa\cdot\partial_{\mathbf{A}})\,\mathbf{L}\mathbf{V}_\infty \tag{40}$$

where,

$$\begin{aligned} \mathcal{I}_\kappa &= \sum_{\alpha=1}^{\kappa-1}\mathcal{N}(\mathbf{V}_{\kappa-\alpha},\mathbf{V}_\alpha) - \sum_{\alpha=1}^{\kappa-1}\mathbf{g}_{\kappa-\alpha+1}\cdot\partial_{\mathbf{A}}\mathbf{L}\mathbf{V}_\alpha, \\ \mathcal{N}(U,V) &= \left(\mathbf{L}\partial_z V E'\right)\partial_x U - \left(\mathbf{L}\partial_x V E'\right)\partial_z U. \end{aligned}$$

By successively solving for different orders we can obtain higher order terms. We will not do the entire solution here as it can be obtained from Arneodo *et al.* (1985). The final amplitude equation with non-linear terms can be written as,

$$\dddot{A} + (\mu_2 - k_3A^2 - k_6A\ddot{A})\ddot{A} + (\mu_1 - k_2A^2 - k_4A\dot{A} - k_5\dot{A}^2)\dot{A} + (\mu_0 - k_1A^2)A = 0 \tag{41}$$

where k_1,k_2,,k_6 are constant coefficients. Their expression can be obtained from Arneodo *et al.* (1985).

Since we want to consider the behavior of the system near the polycriticality point given by eq (23), we can consider asymptotic expansions of form,

$$\begin{aligned} R &= R_0 + \epsilon R_1 + \epsilon^2 R_2 + \ldots, \\ S &= S_0 + \epsilon S_1 + \epsilon^2 S_2 + \ldots, \\ T &= T_0 + \epsilon T_1 + \epsilon^2 T_2 + \ldots. \end{aligned}$$

and scaling given by,

$$\tau = \epsilon t, \qquad A = \epsilon^{3/2} x.$$

Expanding eq(41) using eq(42) and eq(42) to obtain amplitude equations of third order, which are

$$\dddot{x} + \hat{\mu}_2\ddot{x} + \hat{\mu}_1\dot{x} + \hat{\mu}_0 x = k_1 x^3 \tag{42}$$

We have x^5 term in the next order nonlinear terms. We include x^5 in some cases where we need additional fixed points for the system to prevent the solution from blowing up. So the amplitude equation if we include the x^5 term from the higher order nonlinear terms is given as,

$$\dddot{x} + \hat{\mu}_2\ddot{x} + \hat{\mu}_1\dot{x} + \hat{\mu}_0 x = k_1 x^3 - l x^5 \tag{43}$$

This amplitude equation can be rederived for double diffusive convection case where $T = 0$. The equation we will obtain (Coullet & Spiegel (1983)) will be (x^5 nonlinearity included),

$$\ddot{x} + \hat{\mu}_1\dot{x} + \hat{\mu}_0 x = k_1 x^3 - l x^5 \tag{44}$$

We intend to study the effect of the non-normal operator on this system looking for possibility of going out of basin of attraction of a stable fixed point of the system with small initial perturbations. We will also study the effect of noise on this system and possibility of chaos due to presence of small amplitude noise.

6 Results

The second order amplitude equation with A^5 nonlinearity (eq (44)) has five fixed points, given by

$$A = 0, \quad A = \pm\left[\frac{k_1 + \sqrt{k_1^2 - 4l\mu_0}}{2l}\right]^{1/2}, \quad A = \pm\left[\frac{k_1 - \sqrt{k_1^2 - 4l\mu_0}}{2l}\right]^{1/2} \tag{45}$$

The bifurcation diagram for these set of equations is shown in fig 4 for $\mu_1 = 0.71$, $k_1 = 1$ and $l = 0.5$. The solid lines indication the stable solution while the dashed lines indicate the unstable solution.

We have used 4th order Runge-Kutta Scheme to integrate the ODE's. We have used constant time step. The time step was chosen to be $\Delta t = 0.001$ for all simulations discussed here. This time step is small and integration converges for all cases discussed.

6.1 Basin of attraction of origin

Basin of attraction is defined as the region of states, in a dynamical system, around a particular stable steady state, that lead to trajectories going to the stable steady state. We are studying the basin of attraction of the origin.

Fig 5 shows the basin of attraction for eq (44) for the stable fixed point at origin. The basin of attraction is shaped like a slit. It does not close as shown in the figure but extends till infinity. As the figure was calculated numerically it gets truncated.

We have observed that due to non-normal nature of eq (44) even if we give a small initial perturbation inside the basin of attraction the amplitude grows and goes close to the boundary of basin of attraction before decaying back to origin. This suggested that in presence of small noise the system could go away from origin even with very small initial disturbances.

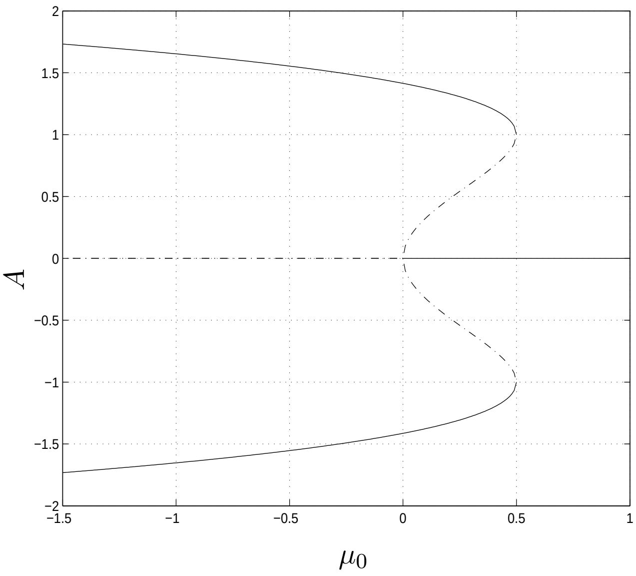

Figure 4: Bifurcation diagram for the amplitude equation. Solid line represents stable solution and dashed line is unstable solution.

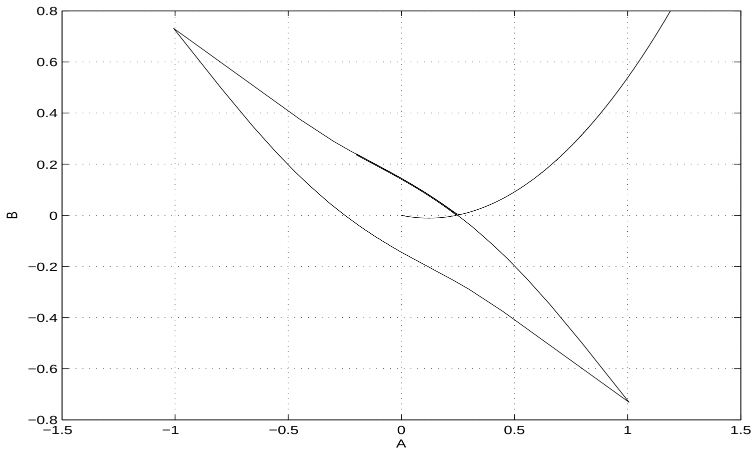

Figure 5: The basin of convergence of Origin for second order amplitude equation with $\mu_1 = 0.5$, $\mu_0 = 0.006$, $k_1 = 1$ and $l = 0$. Two trajectories are shown. One starts just inside the basin and ends up at the origin, while the second trajectory starts at a point just outside the basin and goes to infinity.

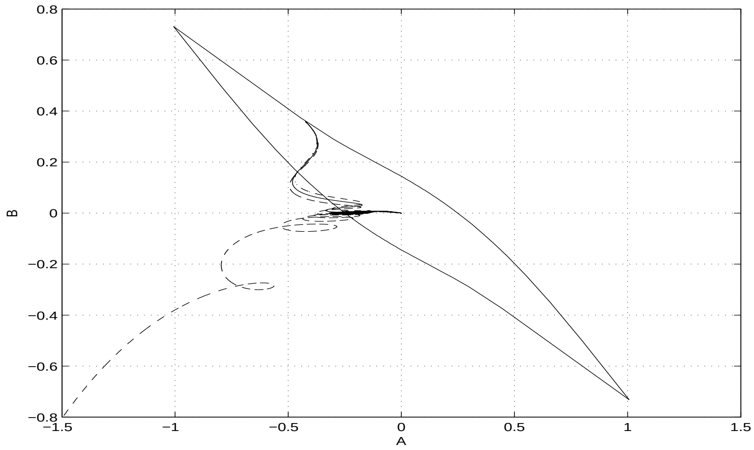

Figure 6: Even small amplitude noise can make big difference. We show the system with $\mu_1 = 0.7$, $\mu_0 = 0.1$, $k_1 = 1$, $l = 0$ and $\epsilon = 10^{-3}$. Three different trajectories are shown for $\omega = 2.4$ (dashed), 2.55 (solid) and 2.7 (dashed again). All have same initial perturbation. This illustrates the importance for small amplitude disturbances in the system.

6.2 Small amplitude noise system

In any physical system background noise of small amplitude is always present. In present case we have only considered marginally stable modes but the stable modes are still in the background and can be considered as background noise. Thus it is important to understand how system behaves in presence of small amplitude noise.

It would be very difficult to analyze the system with random noise. So we considered putting noise of particular frequency. This physically can be explained as the most dominant frequency in the noise. Ultimately we would like to consider the effect of random noise as well.

The modified equations with the noise for triple diffusive system can be written as,

$$\dddot{A} + \hat{\mu}_2\ddot{A} + \hat{\mu}_1\dot{A} + \hat{\mu}_0 A = k_1 A^3 - lA^5 + \epsilon A \sin(\omega t) \tag{46}$$

and for double diffusive system is,

$$\ddot{A} + \hat{\mu}_1\dot{A} + \hat{\mu}_0 A = k_1 A^3 - lA^5 + \epsilon A \sin(\omega t) \tag{47}$$

where, ϵ is atleast an order of magnitude smaller than anything else.

Figure 6 shows three different trajectories for $\omega = 2.4$, 2.55 and 2.7. A frequency of 2.4 makes the system unstable even though we started well within the basin of attraction of the unperturbed equations. For $\omega = 2.55$, we see the system go into an almost periodic orbit for a very long time before decaying back to origin. Figure 7 shows the behavior of A with time for this case. It is clear from this figure that we go into a orbit very close to periodic orbit for $\omega = 2.55$. The system decays down to origin for $\omega = 2.7$. It is clear the we can go to other states even after starting in the basin of attraction of origin with small noise in the system.

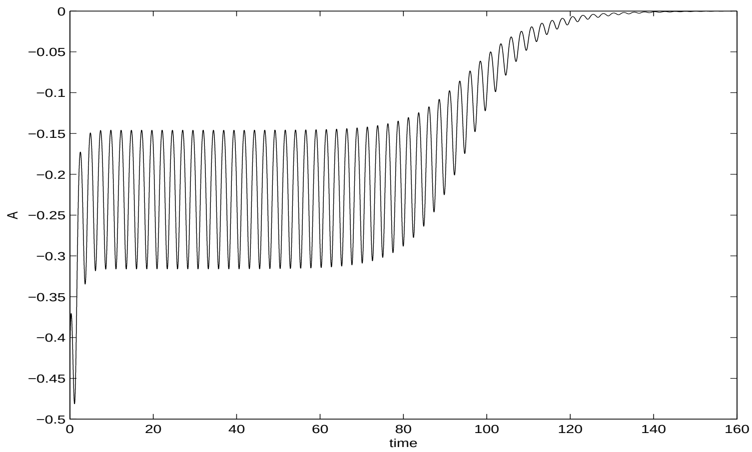

Figure 7: Nearly periodic orbit for $\omega = 2.55$ till $t \approx 80$ We can make system stay on this orbit longer by choosing a more suitable frequency.

It is also possible to go to infinity even if we start infinitesimally close to origin as shown in figure 8. There is range of ω values that can take the system out of the basin of attraction even with very small initial disturbances. This physically seems to suggest that background disturbances even of very small magnitude can make system unstable in linearly stable region.

We get a range of interesting behavior if we take $l \neq 0$ in eq 47. We started increasing ω from 0. For very small values, $\omega \leq 1.1$, almost no effect of the noise is seen on the system and for small initial disturbances the system finally goes to stable fixed point at origin. As ω is increased further the system goes away from origin into whats seems like a chaotic orbit as seen in fig 9.

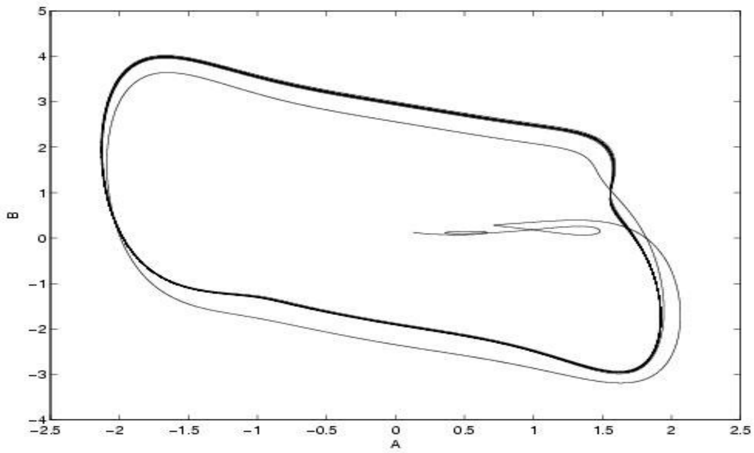

Figure 10: For $\omega = 1.9$ seems to go into a periodic orbit.

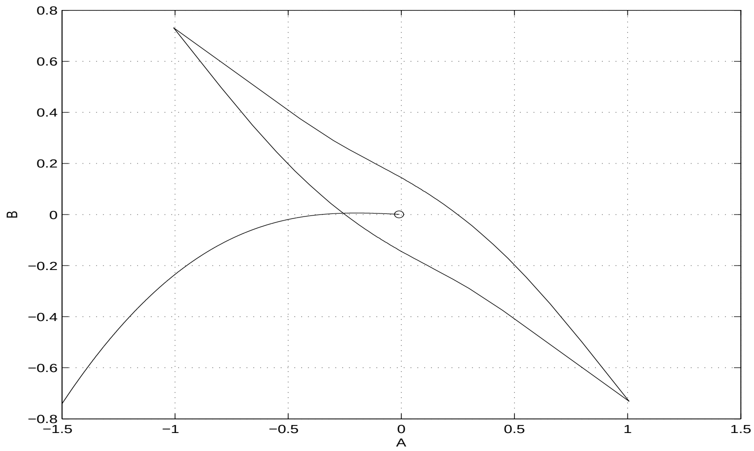

Figure 8: Initial disturbance was of the order of 10^{-3} and still we find a range of frequency of small amplitude noise that takes the system to infinity As we show here with $\mu_1 = 0.71$, $\mu_0 = 0.1$, $\epsilon = 10^{-3}$ and $\omega = 0.01$.

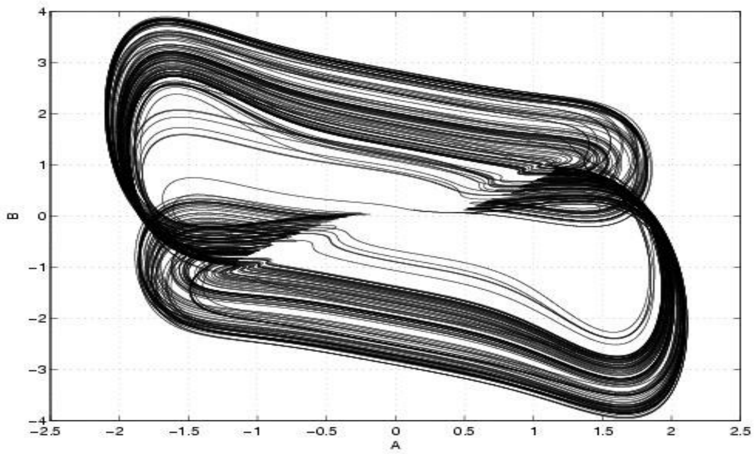

Figure 9: The system with $\mu_1 = 0.5$, $\mu_0 = 0.1$, $\epsilon = 10^{-3}$ and $l = 0.5$. The system above with $\omega = 1.2$ seems to go into a chaotic looking orbit.

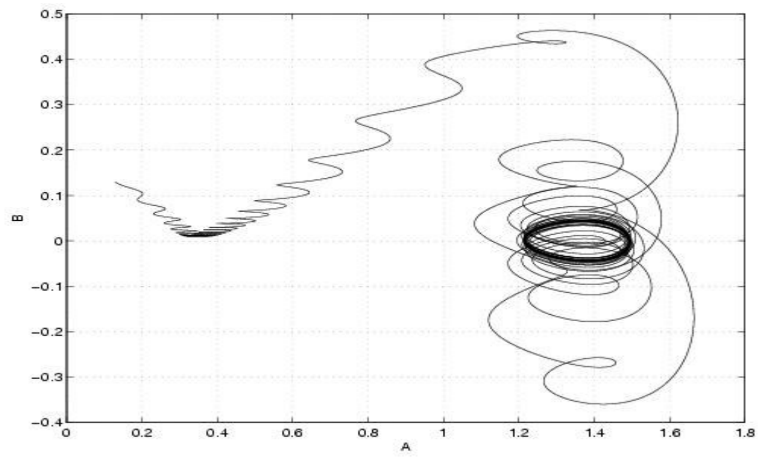

Figure 11: For $\omega = 10.0$ seems to go into a periodic orbit around the outer fixed point.

When the frequency is increased further we observe there is a small range of frequency that again take it to origin. Further increasing the frequency takes the system into a periodic orbit around the outer stable fixed point. This orbit is stable and system stays in this periodic orbit as shown in fig 10.

Increasing frequency further, we go in a regime where the system goes into periodic orbit around the outer fixed point. This periodic orbit is also stable, shown in figure 11. If the frequency is still increased further, the system goes to origin again. Further increase in frequency does not lead to any further change in this behavior.

7 Conclusion

Non-normal growth is observed for double and triple diffusion convection near the poly-critical surface. This suggests that apart from shear flows non-normal growth can also be important in double diffusive convection as a pathway to turbulence, as suggested in Trefethen *et al.* (1993).

Non-normal growth along with the bootstrapping mechanism is not sufficient to take our system away from stable fixed points. We thus considered the possibility of noise along with the bootstrapping mechanism as considered in Gebhardt & Grossmann (1994). We have clearly demonstrated that small noise along with the non-normal nature of system leads to possible escape route out of the basin of attraction of stable fixed point for double diffusive systems.

We observe a range of behavior exhibited by the system as the frequency of the noise term is varied. This behavior appears similar to stochastic resonance in nonlinear systems (c.f. Locher *et al.* (1998)). We are able to observe range of frequencies that can take the system away from stable fixed points. We also observe that a very small disturbance with correct noise frequency can cause the system to go away from the basin of attraction. This supports the suggestions that rapid changes in ocean circulation may indeed be possible even with small disturbances to current environment (c.f. Wiesenfeld & Moss (1995) and

Ganopolski & Rahmstorf (2002)).

Results suggest that small noise have important effect even in linearly stable regime in non-normal systems as non-normality amplifies the effect of noise. Further work is need to consider effect of white noise and possible obtain a probability distribution of escape from basin for different noise levels.

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